Journal of Lie Theory
Volume **20** (2010) 673–687

Principal Basis in Cartan Subalgebra

Rudolf Philippe Rohr

Communicated by B. Ørsted

Abstract. Let $\mathfrak{g}$ be a simple complex Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. In this article we explain how to obtain the principal basis of $\mathfrak{h}$ starting form a set of generators $\{p_1, \cdots, p_r\}, r = \mathrm{rank}(\mathfrak{g})$, of the invariants polynomials $S(\mathfrak{g}^*)^{\mathfrak{g}}$. For each invariant polynomial p, we define a G-equivariant map Dp form $\mathfrak{g}$ to $\mathfrak{g}$. We show that the Gram-Schmidt orthogonalization of the elements $\{Dp_1(\rho^\vee), \cdots Dp_r(\rho^\vee)\}$ gives the principal basis of $\mathfrak{h}$. Similarly the orthogonalization of the elements $\{Dp_1(\rho), \cdots, Dp_r(\rho)\}$ produces the principal basis of the Cartan subalgebra of $\mathfrak{g}^\vee$, the Langlands dual of $\mathfrak{g}$.
Mathematics Subject Classification 2000: 17B.
Key Words and Phrases: Lie algebra, Cartan subalgebra, principal basis, Langlands dual.

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebras over $\mathbf{C}$, let $\mathfrak{h}$ be a choice of the Cartan subalgebra, and let $\{\alpha_1, \cdots, \alpha_r\}$ be a choice for the set of simple roots.. We consider the principal sl_2-triple, $\mathfrak{s}_0 =< e_0, f_0, h_0 >_{\mathbf{C}}$, given by (Lemma 5.2 of [6])

the semi simple element :	$h_0 = 2\rho^\vee$ =the sum of positive coroots
the positive nilpotent element :	$e_0 = e_1 + \cdots + e_r$ =the sum of positive nilpotent elements corresponding to simple roots
the negative nilpotent element :	f_0 such that the well know commutation relation are satisfied, i.e. $[h_0, e_0] = 2e_0, [h_0, f_0] = -2f_0$ and $[e_0, f_0] = h_0$.

The restriction of the adjoint action of $\mathfrak{g}$ to $\mathfrak{s}_0$ gives a representation of $\mathfrak{s}_0$ on $\mathfrak{g}$. Let

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_r \tag{1}$$

be the decomposition of this representation into irreducible $\mathfrak{s}_0$-modules. There is exactly $r = \mathrm{rank}(\mathfrak{g})$ irreducible submodules and their dimensions are $\dim(V_i) =$

ISSN 0949–5932 / $2.50

$2k_i+1$, where the $\{k_i\}_{i=1}^n$ are the exponents of $\mathfrak{g}$ ([6], Section 5). Without loss of generality, we can suppose $1 = k_1 \le k_2 \le \cdots \le k_r$. Each of the modules V_i has a one dimensional intersection with $\mathfrak{h}$, i.e. for each i there exists h_i such that $< h_i >_{\mathbf{C}} = V_i \cap \mathfrak{h}$. By definition ([1], Section 7.2) these elements form the principal basis of $\mathfrak{h}$,

$$\text{principal basis} = \{h_1, \cdots h_r\}.$$

The principal basis was introduce by Kostant to solve a question about Clifford algebra (see remark 4.9).

If all exponents are distinct, the principal basis is uniquely defined (up to constant multiples). This is the case for all simples Lie algebras except for $so(2l, \mathbf{C})$ with l even. In this case there are exactly two exponents with the value $l-1$. Then in the decomposition (1) there are two modules of the same dimension. Hence the definition of the principal basis has to be refined in this case (see 5).

Let $\{\tilde{p}_1, \cdots, \tilde{p}_r\}$ be a choice of homogeneous generators in the ring of invariant polynomials $S(\mathfrak{g}^*)^{\mathfrak{g}}$. We suppose $\deg(p_1) \le \cdots \le \deg(p_r)$. Let $\{e_i^*\}_i$ be an orthonormal basis of $\mathfrak{g}^*$ with respect to a choice of an invariant bilinear form B. We define

$$d\tilde{p}_i(x) = \sum_j \frac{\partial \tilde{p}_i}{\partial e_i^*}(x) \otimes e_i^* \qquad x \in \mathfrak{g}$$

as a G-equivariant map $d\tilde{p}_i : \mathfrak{g} \to \mathfrak{g}^*$, and we consider the set

$$\{d\tilde{p}_1(\rho^\vee), \cdots, d\tilde{p}_r(\rho^\vee)\}$$

of elements of $\mathfrak{g}^*$. Using B we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and we can consider these elements as the elements in $\mathfrak{g}$. Our first result is (Theorem 4.3)

1. The elements $\{d\tilde{p}_1(\rho^\vee), \cdots, d\tilde{p}_r(\rho^\vee)\}$ form a basis of $\mathfrak{h}$.

2. After orthogonalization we obtain the principal basis of $\mathfrak{h} \subset \mathfrak{g}$.

Let ρ be the half sum of positive roots. Using B to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ we construct the set

$$\{d\tilde{p}_1(\rho), \cdots, d\tilde{p}_r(\rho)\}$$

of elements of $\mathfrak{g}^*$. Our second result is (Theorem 4.8)

1. The elements $\{d\tilde{p}_1(\rho), \cdots, d\tilde{p}_r(\rho)\}$ form a basis of $\mathfrak{h}^\vee$, the Cartan subalgebra of $\mathfrak{g}^\vee$, the Langlands dual of $\mathfrak{g}$.

2. After orthogonalization we obtain the principal basis of $\mathfrak{h}^\vee \subset \mathfrak{g}^\vee$.

In section 2 we discuss the relation between roots ans coroots and between the rings $S(\mathfrak{g}^*)^{\mathfrak{g}}$, $S(\mathfrak{h}^*)^{\mathcal{W}}$, $S((\mathfrak{g}^\vee)^*)^{\mathfrak{g}^\vee}$ and $S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee}$.

In section 3 we define the principal three dimensional subalgebra and the principal basis.

Section 4 contains the proof of the first two results, Theorem 4.3 and 4.8.

In section 5 we study in detail the case of $so(2l,\mathbf{C})$ (l even). This Lie algebra has two exponents with the same value, and the principal basis defined by the decomposition (1) is not unique. Let σ be the automorphism of order two of the Dinkin diagram of type D_l (l even). If we require that the vectors of the principal basis be eigenvectors of σ, then the principal basis is unique (up to constant multiple) and orthogonal (Theorem 5.1).

Acknowledgments I would like to thank Anton Alekseev for very helpful discussions. This work was supported in part by the Swiss National Science Foundation.

2. Dual root system

Throughout this section $\mathfrak{g}$ denotes a simple Lie algebra over the field of complex numbers, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\mathcal{R}$ the root system and B a non-degenerate invariant bilinear form on $\mathfrak{g}$, e.g. the Killing form or the canonical form.

The bilinear form B induces an isomorphism of vectors spaces between $\mathfrak{g}$ and $\mathfrak{g}^*$,

$$B^\natural : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}^*,$$

and its inverse

$$B^\flat : \mathfrak{g}^* \xrightarrow{\cong} \mathfrak{g}.$$

They are given by $B^\natural(x) = B(x,\cdot)$ and $B^\flat(\alpha) = x_\alpha$, where x_α is the unique element of $\mathfrak{g}$ such that $B(x_\alpha,\cdot) = \alpha$. As the restriction of B to the Cartan subalgebra $\mathfrak{h}$ is non degenerate, the restriction of $B^\natural$ and $B^\flat$ to the Cartan subalgebra are isomorphisms. Moreover B induces a non degenerate bilinear form on $\mathfrak{h}^*$ which will by denoted again by B and it is given for all $\alpha,\beta \in \mathfrak{h}^*$ by

$$B(\alpha,\beta) = B(B^\flat(\alpha), B^\flat(\beta)).$$

To each root $\alpha \in \mathfrak{h}^*$ we associate a coroot $\alpha^\vee = \gamma(\alpha) \in \mathfrak{h}$,

$$\mathfrak{h}^* \ni \alpha \xrightarrow{\gamma} \alpha^\vee = \frac{2}{B(\alpha,\alpha)} B^\flat(\alpha) \in \mathfrak{h}. \tag{2}$$

The dual roots play the role of the roots in the dual root system $\mathcal{R}^\vee$ of $\mathfrak{g}$. The map γ defines an isomorphism in the sense that $\gamma : \mathcal{R} \xrightarrow{\cong} \mathcal{R}^\vee$. Note that $\mathcal{R}^\vee \subset \mathfrak{h}$. The map γ is canonical, i.e. it is independent of the bilinear form B, but it does not extend to a vector space isomorphism between $\mathfrak{h}^*$ and $\mathfrak{h}$.

Now we will recall the definition of the Langlands dual $\mathfrak{g}^\vee$ of $\mathfrak{g}$.

The Langlands dual $\mathfrak{g}^\vee$ is the Lie algebra whose root system is $\mathcal{R}^\vee$.

This definition is also equivalent to:

The Cartan matrix of $\mathfrak{g}^\vee$ is the transpose of the Cartan matrix of $\mathfrak{g}$.

The Dynkin diagram of $\mathfrak{g}^\vee$ is that of $\mathfrak{g}$ with arrow reversed.

The Langlands dual of the Langlands dual of $\mathfrak{g}$ is $\mathfrak{g}$. For the root systems of simple Lie algebras we have the $X_n^\vee = X_n$ for $X = A, D, E, F, G$ and $B_n^\vee = C_n$.

As $\mathcal{R}$ spans $\mathfrak{h}^*$, dual to the Cartan subalgebra of $\mathfrak{g}$, the dual root system $\mathcal{R}^\vee$ spans $(\mathfrak{h}^\vee)^*$, dual to the Cartan subalgebra of $\mathfrak{g}^\vee$. We have the following identifications

$$\mathfrak{h}^\vee = \mathfrak{h}^* \xleftrightarrow{\cong \text{ by } B} \mathfrak{h} = (\mathfrak{h}^\vee)^*.$$

Note that the left and right identifications are canonical, but the middle one depends on B.

Let $\mathcal{W}$ and $\mathcal{W}^\vee$ be the Weyl groups of $\mathfrak{g}$ and $\mathfrak{g}^\vee$ respectively. Let $\sigma_\alpha \in \mathcal{W}$ and $\sigma_{\alpha^\vee} \in \mathcal{W}^\vee$ be reflections by root α and coroot $\alpha^\vee$ respectively. We have that $B^\flat$ intertwine them, i.e.

$$B^\flat \circ \sigma_\alpha = \sigma_{\alpha^\vee} \circ B^\flat.$$

The isomorphism $B^\flat$ extends to a graded algebra isomorphism

$$B^\flat : S(\mathfrak{h}^*) \xrightarrow{\cong} S((\mathfrak{h}^\vee)^*),$$

and because it intertwines the action of the Weyl groups, its restriction to the Weyl group invariants is a graded algebra isomorphism

$$B^\flat : S(\mathfrak{h}^*)^{\mathcal{W}} \xrightarrow{\cong} S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee}.$$

Using the famous result [3] of Chevalley we have that $S(\mathfrak{h}^*)^{\mathcal{W}}$ and $S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee}$ are generated by $r = \operatorname{rank}(\mathfrak{g})$ homogeneous linearly independent polynomials. Let $\{p_1, \cdots p_r\}$ be a choice of generators of $S(\mathfrak{h}^*)^{\mathcal{W}}$. Then $\{p_1^\flat = B^\flat(p_1), \cdots, p_r^\flat = B^\flat(p_r)\}$ are generators of $S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee}$, i.e.

$$S(\mathfrak{h}^*)^{\mathcal{W}} = \mathbf{C}[p_1, \cdots, p_r] \overset{\cong}{\to} S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee} = \mathbf{C}[p_1^\flat, \cdots, p_r^\flat], \tag{3}$$

Moreover the p_i's (and also the $p_i^\flat$'s) are homogeneous of degree $k_i + 1$, where the integers $\{k_i\}_{i=1}^r$ are the exponents of $\mathfrak{g}$. Note that $\mathfrak{g}$ and $\mathfrak{g}^\vee$ have the same exponents. For simple Lie algebra we can choice these generators so that $1 = k_1 \leq k_2 \leq \cdots \leq k_r$ and the p_1 is given by the restriction of B to $\mathfrak{h}^*$.

Remark 2.1. Using the Tables of [2], we notice that for a simple Lie algebra its exponents are all different, except for simple Lie algebras with roots system of type D_l with l even and greater or equal to 4.

Let Chev : $S(\mathfrak{g}^*) \to S(\mathfrak{h}^*)$ be the Chevalley projection. Its restriction to the invariants induces a graded algebra isomorphism (see [4], Chapter 7, Section 3), i.e Chev : $S(\mathfrak{g}^*)^{\mathfrak{g}} \overset{\cong}{\to} S(\mathfrak{h}^*)^{\mathcal{W}}$. This isomorphism implies that $S(\mathfrak{g}^*)^{\mathfrak{g}}$ and $S((\mathfrak{g}^\vee)^*)^{\mathfrak{g}^\vee}$

are generated by r linearly independent polynomials. We obtain the following commutative diagram of graded algebra isomorphisms :

$$\begin{tikzcd} S(\mathfrak{g}^*)^{\mathfrak{g}} = \mathbf{C}[\tilde{p}_1, \cdots, \tilde{p}_r] \arrow[r, "\cong"] \arrow[d, "\mathrm{Chev}"'] & S((\mathfrak{g}^\vee)^*)^{\mathfrak{g}^\vee} = \mathbf{C}[\hat{p}_1, \cdots, \hat{p}_r] \arrow[d, "\mathrm{Chev}"] \\ S(\mathfrak{h}^*)^{\mathcal{W}} = \mathbf{C}[p_1, \cdots, p_r] \arrow[r, "B^\flat"] & S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee} = \mathbf{C}[p_1^\flat, \cdots, p_r^\flat] \end{tikzcd} \tag{4}$$

Moreover we can choose the generators so that :

1. $\mathrm{Chev}(\tilde{p}_i) = p_i$ and $\mathrm{Chev}(\hat{p}_i) = p_i^\flat$,
2. the upper horizontal isomorphism sends the generators $\tilde{p}_i$ to the generators $\hat{p}_i$.

Note that $\tilde{p}_1$ (resp. $\hat{p}_1$) are defined up to a constant by an invariant bilinear form of $\mathfrak{g}$ (resp. $\mathfrak{g}^*$).

3. Decomposition under action of the principal three dimensional subalgebra, and the principal basis of the Cartan subalgebras

Let $\mathfrak{s} =< e, f, h >_{\mathbf{C}}$ be a sl_2-triple, [1] of a simple complex Lie algebra $\mathfrak{g}$. The restriction of the adjoint representation of $\mathfrak{g}$ to $\mathfrak{s}$ give a representation of $\mathfrak{s}$ on $\mathfrak{g}$, i.e.

$$\mathfrak{s} \ni x \to \mathrm{ad}_x \in End(\mathfrak{g}).$$

is a Lie algebra homomorphism. Note that sl_2-triple exist, indeed every nilpotent elements of $\mathfrak{g}$ can by a Morosov's Theorem embedded in a TDS (see Section 3 of [6]). We consider the decomposition of $\mathfrak{g}$ into a direct sum of the irreducible $\mathfrak{s}$-modules

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_n. \tag{5}$$

Without loss of generality we suppose that $\dim(V_1) \leq \dim(V_2) \leq \cdots \leq \dim(V_n)$. In general the number n of irreducible $\mathfrak{s}$-modules is greater that the rank of $\mathfrak{g}$ (see [6], Section 5), and if its equal to r then $\mathfrak{s}$ will be call principal sl_2-triple (Theorem 5.2 in [6]).

We define a particular principal sl_2-triple, $\mathfrak{s}_0 =< h_0, e_0, f_0 >_{\mathbf{C}}$, with

$$h_0 = 2\rho^\vee, \qquad e_0 = \sum_i e_i \quad \text{and} \quad f_0 = \sum_i c_i f_i, \tag{6}$$

where $\rho^\vee$ is the half sum of positive coroots, e_i (rep. f_i) are the root vectors corresponding to the simple roots α_i (resp. $-\alpha_i$) such that $B(e_i, f_i) = 1$ and the c_i are given by the relation $\sum_i c_i B^\flat(\alpha_i) = 2\rho^\vee$. Moreover each principal sl_2-triple is conjugate to (6), and if we require the semisimple element to be in $\mathfrak{h}$, then the principal sl_2-triple is given by (6). It is of course possible to multiply each e_i by

[1] The commutation relations are $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$.

a scalar and then divide f_i by the same scalar.

Using $\mathfrak{s}_0$ in the decomposition (5) we have that $n = \text{rank}(\mathfrak{g})$ and each V_i is an irreducible $\mathfrak{s}_0$-module. Moreover $\dim(V_i) = 2k_i + 1$, where the $\{k_i\}_{i=1}^r$ are the exponents of $\mathfrak{g}$. This facts implies that for all i the intersection of V_i and $\mathfrak{h}$ is one dimensional, i.e.

for all i, there exist $h_i \in \mathfrak{h}$ such that $< h_i >_{\mathbf{C}} = V_i \cap \mathfrak{h}$.

These elements form a basis of $\mathfrak{h}$. This basis is called the principal basis of $\mathfrak{h}$ ([1], Section 7),

$$\text{principal basis of } \mathfrak{h} = \{h_1, \cdots, h_r\}. \tag{7}$$

For $k_i \neq k_j$ we have that h_i is orthogonal to h_j relatively to B. Indeed, we have that $ad_{e_0}ad_{f_0}(h_i) = \frac{k_i(k_i+1)}{2}h_i$, then using the invariance of B we conclude that for $k_i \neq k_j$, h_i is orthogonal to h_j. Hence if all exponents are distinct, up to a constant, the principal basis is unique. In fact a similar argument shows that the decomposition (5) under $\mathfrak{s}_0$ is an orthogonal decomposition. The only cases when the are two equal exponent occur for the Lie algebras with the root system of type D_l with l even, i.e. for $so(2l, \mathbf{C})$. We study this case in 5.

Remark 3.1. As vector space $V_1 = \mathfrak{s}_0$, then we have (up to a constant) $h_1 = \rho^\vee$.

Remark 3.2. The elements $\{h_i\}_{i=1}^r$ of the principal basis are characterize by

1. $h_i \in \text{Ker}(ad^{k_i+1}(e_0))$,
2. $h_i \notin \text{Ker}(ad^{k_i}(e_0))$,
3. for $i \neq j$, $h_i \perp h_j$.

Remark 3.3. If an element $h \in \mathfrak{h}$ satisfies $ad^{k_i+1}(e_0)(h) = 0$, then $h \in (V_1 \oplus \cdots \oplus V_i) \cap \mathfrak{h}$.

We will end this section by giving two examples of principal basis.

Example 3.4. Let $\mathfrak{g}$ be the Lie algebra of type G_2. Its Cartan subalgebra $\mathfrak{h}$ is 2-dimensional. Let α_1 and α_2 be a choice of the simples roots. We choose B to be the canonical form, i.e. on simple root it is given by

$$B(\alpha_1, \alpha_1) = 2 \qquad B(\alpha_1, \alpha_2) = -1 \qquad B(\alpha_2, \alpha_2) = \frac{2}{3}.$$

The coroots are then given by

$$\alpha_1^\vee = B^\flat(\alpha_1) \qquad \alpha_2^\vee = 3B^\flat(\alpha_2).$$

The first vector in the principal basis is the sum of the positive coroots, and the second is one orthogonal to the first. They are (up to constant)

$$h_1 = 5\alpha_1^\vee + 3\alpha_2^\vee \qquad h_2 = -3\alpha_1^\vee + \alpha_2^\vee.$$

Example 3.5 (Principal basis for $\mathrm{sl}(3, \mathbf{C})$). Let e_{ij} be the canonical basis of $\mathrm{Mat}(3, \mathbf{C})$. The standard choice of basis for the Cartan subalgebra of $\mathrm{sl}(3, \mathbf{C})$ is $\tilde{h}_1 = e_{11} - e_{22}$ and $\tilde{h}_2 = e_{22} - e_{33}$. The positive (resp. negative) nilpotent elements are given by e_{12}, e_{23} and e_{13} (resp. e_{21}, e_{32} and e_{31}). In terms of this basis the principal sl_2-triple $\mathfrak{s}_0$ is given by $h_0 = 2\tilde{h}_1 + 2\tilde{h}_2$, $e_0 = e_{12} + e_{23}$ and $f_0 = 2e_{21} + 2e_{32}$. In the decomposition (5) the two irreducible $\mathfrak{s}_0$-modules are given by

$$V_1 =< e_0, h_0, f_0 >_{\mathbf{C}},$$

$$V_2 =< e_{13}, e_{23} - e_{12}, \tilde{h}_1 - \tilde{h}_2, e_{32} - e_{21}, e_{31} >_{\mathbf{C}} .$$

Then the principal basis is given by (up to constants)

$$\{h_1 = \tilde{h}_1 + \tilde{h}_2 = e_{11} - e_{33}, h_2 = \tilde{h}_1 - \tilde{h}_2 = e_{11} - 2e_{22} + e_{33}\}.$$

By a direct calculation we verify that it is an orthogonal basis.

4. How to obtain the principal basis of the Cartan subalgebra using the generators of the invariants polynomials

Let $\mathfrak{g}$ be a simple Lie algebra over the complex field, and $\tilde{p} \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ an invariant polynomial. We consider $d\tilde{p}$ the differential of this polynomial. It is defined by

$$d\tilde{p} = \sum_i \frac{\partial \tilde{p}}{\partial e_i^*} \otimes e_i^* \in S(\mathfrak{g}^*) \otimes \mathfrak{g}^*,$$

where $\{e_i^*\}$ is an orthonormal basis of $\mathfrak{g}^*$. We view this differential as an map form $\mathfrak{g}$ to $\mathfrak{g}^*$, i.e.

$$\mathfrak{g} \ni x \longrightarrow d\tilde{p}(x) = \sum_i \frac{\partial \tilde{p}}{\partial e_i^*}(x) \cdot e_i^* \in \mathfrak{g}^*.$$

Let $G = \mathrm{Int}(\mathfrak{g})$ be the group of inner automorphisms of $\mathfrak{g}$. It is generated by the elements of the form e^{ad_x} with $x \in \mathfrak{g}$ nilpotent. The invariant bilinear form B is invariant by G, i.e. for all $g \in G$ and $x, y \in \mathfrak{g}$ we have $B(g \cdot x, y) = B(x, g^{-1} \cdot y)$. Moreover, G intertwines the isomorphisms $B^\flat$ and $B^\natural$.

Proposition 4.1. *The map $d\tilde{p}$ is G-equivariant, i.e. for all $g \in G$*

$$d\tilde{p} \circ g = g \circ d\tilde{p}.$$

Proof. Let $x \in \mathfrak{g}$ and $g \in G$, then $d\tilde{p}(g \cdot x) = \sum_i \frac{\partial \tilde{p}}{e_i}(g \cdot x) e_i$. Let $y_i = g^{-1} x_i$ be a new orthonormal basis of $\mathfrak{g}$. Then using the fact that p is G-invariant we obtain

$$\sum_i \frac{\partial \tilde{p}}{e_i}(g \cdot x) e_i = \sum_i \frac{\partial \tilde{p}}{g \cdot y_i}(g \cdot x) g \cdot y_i = \sum_i \frac{\partial (g \cdot \tilde{p})}{g \cdot y_i}(g \cdot x) g \cdot y_i = \sum_i \frac{\partial \tilde{p}}{y_i}(x) g \cdot y_i.$$

■

Let $D\tilde{p}$ be the composition of $d\tilde{p}$ with $B^\flat$, i.e.

$$D\tilde{p} = B^\flat \circ d\tilde{p} : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Obviously $D\tilde{p}$ is G-equivariant.

We can consider the same construction with an invariant polynomial $p \in S(\mathfrak{h}^*)^{\mathcal{W}}$. Then we obtain the map

$$Dp : \mathfrak{h} \longrightarrow \mathfrak{h}.$$

The following proposition relates these two constructions.

Proposition 4.2. *Let $p \in S(\mathfrak{h}^*)^{\mathcal{W}}$ and $\tilde{p} \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ be invariant polynomials such that $Chev(\tilde{p}) = p$. Then for all $h \in \mathfrak{h}$ we have*

$$Dp(h) = D\tilde{p}(h).$$

Proof. By Lemma 7.3.6 of [4], there exist $a \in S(\mathfrak{g}^*)^{\mathfrak{g}}\mathfrak{n}_+^*$ such that $\tilde{p} = p + a$. But $Da(h) = 0$ because $\mathfrak{h}^*$ is orthogonal to $\mathfrak{n}_+^*$. ■

Let $\{\tilde{p}_1, \cdots, \tilde{p}_r\}$ be a choice of generators of $S(\mathfrak{g}^*)^{\mathfrak{g}}$, with $\deg(p_1) \leq \deg(p_2) \leq \cdots \leq \deg(p_r)$. Let $\rho^\vee \in \mathfrak{h}$ be the half sum of positive coroots. By the preceding proposition the elements

$$\{D\tilde{p}_1(\rho^\vee), \cdots D\tilde{p}_r(\rho^\vee)\} \tag{8}$$

are in $\mathfrak{h}$. Note that up to a constant we can chose $\tilde{p}_1$ such that $D\tilde{p}_1(\rho^\vee) = \rho^\vee$. By Theorem 3 of [8] with $A = \rho^\vee$, this family is linearly independent. Hence it is a basis of $\mathfrak{h}$. Using the Gram-Schmidt orthogonalization process on this basis we obtain the orthogonal basis :

$$\{D\tilde{p}_1(\rho^\vee), \cdots D\tilde{p}_r(\rho^\vee)\} \tag{9}$$

$$\Big\downarrow{\scriptstyle \text{Gram Schmidt}}$$

$$\begin{aligned}\{h_1 &= D\tilde{p}_1(\rho^\vee),\\ h_2 &= D\tilde{p}_2(\rho^\vee) - \lambda_{21} D\tilde{p}_1(\rho^\vee),\\ &\cdots,\\ h_r &= D\tilde{p}_r(\rho^\vee) - \lambda_{r,r-1} D\tilde{p}_1(\rho^\vee) - \cdots\}.\end{aligned}$$

The first result of this section is the following Theorem.

Theorem 4.3. *The orthogonal basis (9) of $\mathfrak{h}$ is (up to constant multiple) the principal basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.*

Before we prove this Theorem we will show how it work in the Examples 3.4 and 3.5.

Example 4.4 (Continuation of Example 3.4). We will compute the principal basis using Theorem 4.3 and then compare with the previous result. Let $\{x, y\}$ be the orthonormal basis of the Cartan subalgebra given by

$$x = \frac{\alpha_1^\vee}{\sqrt{2}} \qquad y = \sqrt{\frac{2}{3}} \left(\alpha_2^\vee + \frac{3}{2}\alpha_1^\vee \right).$$

Let $\{x^*, y^*\}$ be the dual basis. A choice for generators of $S(\mathfrak{h}^*)^{\mathcal{W}}$ is given by

$$p_1 = (x^*)^2 + (y^*)^2 \qquad p_2 = 33(x^*)^6 + 27(y^*)^6 + 45(x^*)^4(y^*)^2 + 135(x^*)^2(y^*)^4.$$

The differentials of these polynomials computed at $\rho^\vee$ give (up to constant multiple)

$$Dp_1\rho^\vee \propto \rho^\vee \qquad Dp_2\rho^\vee \propto 2425\alpha_1^\vee + 1383\alpha_2^\vee.$$

We remark that they are not orthogonal, bur after orthogonalization we obtain the principal basis.

Remark 4.5. The previous example shows that the orthogonalization is indispensable.

Example 4.6 (Continuation of example 3.5). We will compute the principal basis using Theorem 4.3 and then compare with the previous result. A choice of generators of $S(\mathfrak{h}^*)^{\mathcal{W}}$ is given by ([7], Section 3),

$$p_1 = (e_{11}^*)^2 + (e_{22}^*)^2 + (e_{33}^*)^2 \qquad p_2 = (e_{11}^*)^3 + (e_{22}^*)^3 + (e_{33}^*)^3,$$

where $\{e_{11}^*, e_{22}^*, e_{33}^*\}$ is the dual basis of $\{e_{11}, e_{22}, e_{33}\}$. As explain in [7], these polynomials belong to the dual of the Cartan subalgebra of $gl(3, \mathbf{C})$. We work with the basis $\{\tilde{h}_1^* = e_{11}^* - e_{22}^*, \tilde{h}_2^* = e_{22}^* - e_{33}^*\}$ completed by the central element $c^* = e_{11}^* + e_{22}^* + e_{33}^*$. In this basis these polynomials are written as follows:

$$p_1 = (\tilde{h}_1^*)^2 + (\tilde{h}_2^*)^2 + \tilde{h}_1^*\tilde{h}_2^* + (c^*)^2$$

$$p_2 = 2(\tilde{h}_1^*)^3 - 2(\tilde{h}_2^*)^3 + 3(\tilde{h}_1^*)^2\tilde{h}_2^* - 3\tilde{h}_1^*(\tilde{h}_2^*)^2 + \text{ terms with } c^*.$$

Remark that $< \tilde{h}_1^*, \tilde{h}_2^* >_{\mathbf{C}} \perp < c^* >_{\mathbf{C}}$. Their differentials evaluated at $\rho^\vee$ are (up to a constant)

$$Dp_1(\rho^\vee) = \tilde{h}_1 + \tilde{h}_2 \qquad Dp_2(\rho^\vee) = \tilde{h}_1 - \tilde{h}_2.$$

This is fortunately the same basis as before. Note that in this case the choice of polynomial is unique (up to constants) and we do not need to orthogonalize.

Proof. [Proof of Theorem 4.3] This proof works if all exponents are different. This is the case for all simple Lie algebras except for type D_l with l even. We prove this case in 5.

1. Using Theorem 3 of [8] with $A = \rho^\vee$, we conclude that the elements (8) are linearly independents.

2. Let $\mathfrak{s}_0$ be the principal sl_2-triple given by (6). Then $D\tilde{p}_i(\rho^\vee) \in \mathrm{Ker}(\mathrm{ad}_{e_0}^{k_i+1})$. Indeed using the G-equivariance of $D\tilde{p}_i$) (Proposition 4.1) for $e^{t\mathrm{ad}_{e_0}}$ (t a parameter) we have

$$(e^{t\mathrm{ad}_{e_0}} D\tilde{p}_i)(\rho^\vee) = D\tilde{p}_i(e^{t\mathrm{ad}_{e_0}}\rho^\vee) = (D\tilde{p}_i \circ (1 + t\mathrm{ad}_{e_0}))(\rho^\vee),$$

but and $D\tilde{p}_i$ is a polynomial of degree k_i, and the term in t^{k_i+1} is vanishes.

3. The step (b) implies that for all i

$$D\tilde{p}_i(\rho^\vee) \in (V_1 \oplus \cdots \oplus V_i) \cap \mathfrak{h}.$$

4. Then if we orthogonalize the family

$$\{D\tilde{p}_1(\rho^\vee), \cdots, D\tilde{p}_r(\rho^\vee)\}$$

as (9) we obtain (up to constant multiple) the principal basis of $\mathfrak{h}$. ■

This Theorem provides a method to compute the principal basis of the Cartan subalgebra of $\mathfrak{h}$ using a set of generators of the invariants polynomials $S(\mathfrak{g}^*)^{\mathfrak{g}}$. Now we will give the second result of this section, which explain how to obtain the principal basis of the Cartan subalgebra of $\mathfrak{g}^\vee$ the Langlands dual of $\mathfrak{g}$ using a set of generators of $S(\mathfrak{g}^*)^{\mathfrak{g}}$.

As explained before $d\tilde{p}$ is an G-equivariant map from $\mathfrak{g}$ to $\mathfrak{g}^*$. We consider the following map, which is also G-equivariant,

$$\hat{D}\tilde{p} = d\tilde{p} \circ B^\natural : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*.$$

We have the analogue of Proposition 4.2

Proposition 4.7. *Let $p \in S(\mathfrak{h}^*)^{\mathcal{W}}$ and $\tilde{p} \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ be invariant polynomials such that $Chev(\tilde{p}) = p$. Then for all $\lambda \in \mathfrak{h}^*$ we have*

$$\hat{D}p(\lambda) = \hat{D}\tilde{p}(\lambda).$$

The proof is the same as for Proposition 4.2.

The second result of this section is the following theorem.

Theorem 4.8. *Let ρ be the half sum of the positive roots of $\mathfrak{g}$. Let*

$$\{\hat{D}\tilde{p}_1(\rho), \cdots, \hat{D}\tilde{p}_r(\rho)\}$$

be a family of element of $\mathfrak{h}^*$, the Cartan subalgebra of $\mathfrak{g}^\vee$. Then

1. *This family is linearly independent.*

2. *Using the Gram-Schmidt process as in (9) we obtain (up to constant) the principal basis of $\mathfrak{h}^\vee \subset \mathfrak{g}^\vee$.*

Proof.

1. Let $p \in S(\mathfrak{h}^*)^{\mathcal{W}}$ be an invariant polynomial and define $p^\flat = B^\flat(p) \in S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee}$. As for p, we consider $dp^\flat$ to be a map from $\mathfrak{h}^*$ to $\mathfrak{h}$. We have the following relation between dp and $dp^\flat$,
$$dp \circ B^\flat = B^\natural \circ dp^\flat.$$

2. We take $\tilde{p} \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ such that $\mathrm{Chev}(\tilde{p}) = p$. Using the Proposition 4.7 and (a) we have for all $\lambda \in \mathfrak{h}^*$,
$$\hat{D}\tilde{p}(\lambda) = \hat{D}p(\lambda) = (B^\natural \circ dp^\flat)(\lambda).$$

3. We take $\hat{p} \in S((\mathfrak{g}^\vee)^*)^{\mathfrak{g}^\vee}$ such that $\mathrm{Chev}(\hat{p}) = p^\flat$. We use the equalities in (b) to obtain
$$\hat{D}\tilde{p}(\lambda) = \hat{D}p(\lambda) = (B^\natural \circ dp^\flat)(\lambda) = (B^\natural \circ d\hat{p})(\lambda).$$
The proof of the last equality is the same as for the Proposition 4.2 with $\mathfrak{g}^\vee$ at the place of $\mathfrak{g}$.

4. By (c) point we see that the family $\{\hat{D}\tilde{p_1}(\rho), \cdots, \hat{D}\tilde{p_r}(\rho)\}$ is the same as the family $\{(B^\natural \circ d\hat{p_1}^\flat)(\rho), \cdots, (B^\natural \circ d\hat{p_r}^\flat)(\rho)\}$. The result (a) is given by Theorem 3 in [8] and the result (b) by using Theorem 4.3 with $\mathfrak{g}^\vee$ replacing $\mathfrak{g}$.

■

Remark 4.9 (The Kostant conjecture about the principal basis)**.** Let $\mathrm{Cl}(\mathfrak{g}, K)$ be the Clifford algebra of the Lie algebra $\mathfrak{g}$ and K its Killing form. Consider the Harish-Chandra projection $\Phi : \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cl}(\mathfrak{h})$ defined by the decomposition $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} = \mathrm{Cl}(\mathfrak{h}) \oplus \mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+ \cap \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$, where $\mathfrak{n}_+$ are the positives nilpotent elements. In particular Kostant showed that the image of the Harish-Chandra projection of the primitive generators of $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \cong \bigwedge \mathfrak{g}^{\mathfrak{g}}$ are contained in $\mathfrak{h} \subset \mathrm{Cl}(\mathfrak{h})$. Kostant conjectured that the Harish-Chandra projection of the primitive generators of $\mathfrak{g}$ gives the principal basis of the Cartan Subalgebra of the Langlands dual $\mathfrak{g}^\vee$ (here using K, $\mathfrak{h}$ and $\mathfrak{h}^\vee$ are identified). This conjecture was partially proven by Bazlov in his thesis [1].

5. The $so(2l, \mathbf{C})$ case (l even)

Throughout this section $\mathfrak{g} = so(2l, \mathbf{C})$ with l even, $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, $\mathcal{W}$ the Weyl group and B is an invariant bilinear form. The root system of $\mathfrak{g}$ is D_l and the exponents are given by [2], Table IV
$$\{1, 3, 5, \cdots, l-1, l-1, \cdots, 2l-5, 2l-3\},$$
Then in the decomposition of $\mathfrak{g}$ into irreducible $\mathfrak{s}_0$-modules,
$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_{l/2} \oplus V_{l/2+1} \oplus \cdots \oplus V_l,$$

the modules $V_{l/2}$ and $V_{l/2+1}$ have the same dimension $2l-1$.

This decomposition does not determine $V_{l/2}$ and $V_{l/2+1}$ uniquely. Indeed $V_{l/2}$ and $V_{l/2+1}$ are generated by the highest weight vectors v_1 and v_2, i.e. $V_{l/2} = U(\mathfrak{s}_o)v_1$ and $V_{l/2+1} = U(\mathfrak{s}_o)v_2$. But v_1+v_2 and v_1-v_2 also generate two irreducible $\mathfrak{s}_o$-modules whose the direct sum is equal to $V_{l/2}\oplus V_{l/2+1}$, i.e. $U(\mathfrak{s}_o)(v_1+v_2)\oplus U(\mathfrak{s}_o)(v_1-v_2) = V_{l/2}\oplus V_{l/2+1}$.

The following decomposition is uniquely determined by the action of the principal sl_2-triple,

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_{l/2-1} \oplus W \oplus V_{l/2+2} \oplus \cdots \oplus V_l, \tag{10}$$

where $W = V_{l/2}\oplus V_{l/2+1}$. Note that this decomposition is orthogonal relatively to B.

Hence the principal basis is uniquely determined modulo a choice of two linearly independent vectors in $W\cap\mathfrak{h}$, i.e.

$$\text{principal basis of } \mathfrak{h} = \{h_1, \cdots, h_l, h_{l+1}, \cdots, h_r\},$$

with h_i such that $< h_i >_{\mathbf{C}} = V_i\cap\mathfrak{h}$ for all $i \neq l/2, l/2+1$ and $< h_{l/2}, h_{l/2+1} >_{\mathbf{C}} = W\cap\mathfrak{h}$. Moreover if we chose $h_{l/2}$ and $h_{l/2+1}$ to be orthogonal, then the principal basis is uniquely determined modulo a rotation in W.

Now we will give the modifications in the proof of Theorem 4.3.

(a) (b) The first and second step remain true.

(c) The third step is modified as follows: for all $i \neq l/2, l/2+1$ we have

$$D\tilde{p}_i(\rho^\vee) \in (V_1\oplus\cdots\oplus V_i)\cap\mathfrak{h},$$

and for $i = l/2$ or $i = l/2+1$ we have

$$D\tilde{p}_i(\rho^\vee) \in (V_1\oplus\cdots\oplus W)\cap\mathfrak{h}.$$

(d) The fourth step reaming true with a suitable choice of basis vectors in $W\cap\mathfrak{h}$.

Let $\{\alpha_1, \cdots, \alpha_l\}$ be the l simple roots of $\mathfrak{g}$. We define the automorphism σ by

$$\sigma(\alpha_i) = \alpha_i \quad (i = 1, \cdots l-2) \qquad \sigma(\alpha_{l-1}) = \sigma(\alpha_l) \qquad \sigma(\alpha_l) = \sigma(\alpha_{l-1}).$$

This automorphism induces an automorphism on the Lie algebra, see Section 7.9 of [5]. As this automorphism is of order two it is diagonalizable and its eigenvalues are $+1$ and -1. Let $\mathfrak{g} = \mathfrak{g}_0\oplus\mathfrak{g}_1$ be the decomposition in the eigenspace $\mathfrak{g}_0$, resp. $\mathfrak{g}_1$, of eigenvalues $+1$, resp. -1. In [5], Chapter 8, Kac showed that the roots system of $\mathfrak{g}_0$ is B_{l-1}. This implies that the dimension of $\mathfrak{g}_1$ is $2l-1$. Indeed $\dim(D_l) = l(2l-1)$ and $\dim(B_{l-1}) = (l-1)(2l-1)$.

We denote by B the canonical form of $\mathfrak{g}$, i.e. $B(\alpha_i, \alpha_i) = 2$.

Theorem 5.1. *Consider the decomposition (10) of $\mathfrak{g}$, we have*

1. *All the irreducible $\mathfrak{s}_0$-modules V_i are contained in the eigenspace $\mathfrak{g}_0$.*
2. *The $\mathfrak{s}_0$-module W is invariant by σ, i.e. $\sigma(W) = W$.*
3. *The automorphism σ provides the decomposition of W into $W = W_0 \oplus W_1$, where $W_i \subset \mathfrak{g}_i$. Moreover, this is a decomposition into two irreducible $\mathfrak{s}_0$-submodules and they are orthogonal.*

Hence if we require that the elements of the principal basis should be eigenvectors of σ, then the principal basis is uniquely determined (up to constants).

The proof of this Theorem follows from the next three propositions.

Proposition 5.2. *Let $\mathfrak{s}_0 =< e_0, f_0, h_0 >$ be the principal sl_2-triple. Then its adjoint action commute with the automorphism σ, i.e.*

$$\sigma \circ ad_{e_0} = ad_{e_0} \circ \sigma \qquad \sigma \circ ad_{f_0} = ad_{f_0} \circ \sigma \qquad \sigma \circ ad_{h_0} = ad_{h_0} \circ \sigma.$$

Proof. The principal TDS $\mathfrak{s}_0$ is given by

$$h_0 = 2(l-1)\alpha_1^\vee + 2(2l-3)\alpha_2^\vee + \cdots + (l-2)(l+1)\alpha_{l-2}^\vee + l(l-1)/2(\alpha_{l-1}^\vee + \alpha_l^\vee),$$

$$e_0 = e_1 + \cdots e_l,$$

$$f_0 = 2(l-1)f_1 + \cdots + (l-2)(l+1)f_{l-2} + l(l-1)/2(f_{l-1} + f_l),$$

where e_i is the positive nilpotent element corresponding to the simple root α_i, and the f_i is the corresponding negative nilpotent elements such that $B(e_i, f_i) = 1$. Clearly they are eigenvectors of eigenvalues $+1$. The result follows from the equation

$$\sigma(ad_{e_0}x) = ad_{\sigma(e_0)}\sigma(x) = ad_{e_0}\sigma(x).$$

The proof for h_0 and f_0 is similar. ■

Proposition 5.3. *Let $V \in \mathfrak{g}$ be a irreducible $\mathfrak{s}_0$-module. If there exist $x \in V$ such that $\sigma(x) = x$ (or $\sigma(x) = -x$), then $V \subset \mathfrak{g}_0$ (or $V \subset \mathfrak{g}_1$).*

Proof. Let $x \in V$ be a eigenvector of eigenvalue λ for σ ($\lambda = \pm 1$). Then using Proposition 5.2, we have that all vectors $y \in V = U(\mathfrak{s}_0)x$ are eigenvectors of eigenvalue λ. ■

Before the next proposition we introduce the standard choice for the generators of $S(\mathfrak{h}^*)^{\mathcal{W}}$.

Let $\{e_1^*, \cdots e_l^*\}$ be the standard orthonormal basis of $\mathfrak{h}^*$. The simple roots are given by

$$\alpha_1 = e_1^* - e_2^* \qquad \cdots \qquad \alpha_{l-1} = e_l^* - e_{l-1}^* \qquad \alpha_l = e_l^* + e_{l-1}^*.$$

Recall that on the elements of the basis the canonical bilinear invariant form is given by $B(e_i^*, e_j^*) = \delta_{ij}$. In term of this basis the standard generators are given by ([7], Section 3)

$$p_i = \sum_j (e_j^*)^{2i} \quad i = 1, \cdots, l-1 \qquad \text{and} \qquad p_e = e_1^* e_2^* \dots e_l^*.$$

The half sum of positive roots is given by

$$\rho^\vee = (l-1)(e_1) + (l-2)(e_2) + \cdots + e_{l-1},$$

where $\{e_1, \cdots, e_l\}$ is the dual basis of $\{e_1^*, \cdots, e_l^*\}$.Note that the two polynomials of degree l are p_e and $p_{l/2}$.

The differential of these generators computed at $\rho^\vee$ are,

$$\begin{array}{rc}
Dp_1(\rho^\vee) = & 2\rho^\vee \\
Dp_2(\rho^\vee) = & 4 \cdot 2^3((l-1)^3 e_1 + \cdots + e_{l-1}) \\
\cdots & \\
Dp_e(\rho^\vee) = & l! \cdot 2^{l-1} e_l \\
Dp_{l/2}(\rho^\vee) = & \cdots \\
\cdots & \\
Dp_{l-1}(\rho^\vee) = & (2l-2)2^{2l-3}((l-1)^{2l-3} e_1 + \cdots + e_{l-1})
\end{array}$$

We remark that $< Dp_1(\rho^\vee), \cdots, Dp_{l-1}(\rho^\vee) >_{\mathbf{C}}$ is orthogonal to $< Dp_e(\rho^\vee) >_{\mathbf{C}}$.

From these calculations we deduce the next proposition

Proposition 5.4. *We have that $Dp_i(\rho^\vee) \in \mathfrak{g}_0$ for all $i = 1, \cdots, l-1$ and $Dp_e(\rho^\vee) \in \mathfrak{g}_1$.*

Now we prove the Theorem 5.1.

Proof. [Proof of Theorem 5.1]

1. We orthogonalize the basis (11) and obtain the principal basis $\{h_1, \cdots, h_l\}$. Note that all h_i except $h_{l/2}$ are in the space $< e_1, \cdots, e_{l-1} >_{\mathbf{C}}$ and $h_{l/2} \in < e_l >_{\mathbf{C}}$. By Proposition 5.3, this proves the first assertion.

2. The second assertion follows from Proposition 5.3 and the fact that $W = U(\mathfrak{s}_0) < h_{l/2}, h_{l/2+1} >_{\mathbf{C}}$.

3. For the third assertion we have that $W_0 = U(\mathfrak{s}_0) h_{l/2+1}$ and $W_1 = U(\mathfrak{s}_0) h_{l/2}$. ■

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Rudolf Philippe Rohr
Department of Mathematics
University of Geneva
2-4 rue de Lievre
C.P.64, 1211 Geneva 4, Switzerland
Rudolf.Rohr@unige.ch

Received April 30, 2008
and in final form July 28, 2010