# Whom to Trust in a Signed Network? Optimal Solution and two Heuristic Rules

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**Abstract**

Signed networks can capture positive and negative relationships in a social network. We assume that the network nodes are of two different types and link signs correlate with the node types, which induces some patterns of structural balance. Given a signed network and information on the type of some source nodes, we consider an observer outside the network who attempts to judge the type of a given target node. Computing the globally optimal belief by Bayes' rule involves considering exponentially many states. We propose a much simpler heuristic that is based on the shortest paths between source nodes and target nodes. Theoretically, this heuristic is weakly better than another heuristic from the literature and it coincides with the Bayesian rule when the shortest paths between the source nodes and a target node are unique and non-overlapping. With simulations, we assess the accuracy of the three rules and find that differences can be substantial. The shortest path heuristic is better than the other heuristic in handling multiple source nodes, even though it aggregates information suboptimally. The crucial network statistic for accuracy is the average distance in a network.

Keywords: signed graphs, social networks, trust, structural balance, learning

## 1 Introduction

The extensive study of social networks across several disciplines has brought to light many patterns of how people (or organizations) are related to each other. While in most studies network ties are assumed to be positive, e.g. friendship; negative ties, e.g. hostility, also

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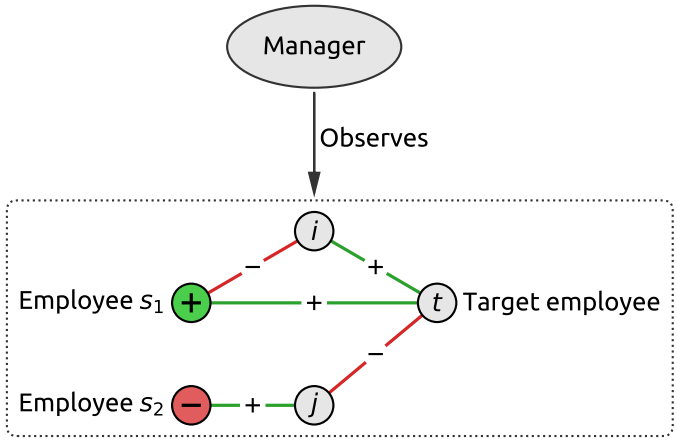


Figure 1: Example: A manger learns about qualities of a target employee $t$ from observing the positive and negative relations that this employee holds.

exist in many applications. Signed networks capture both positive and negative ties.[1] When using network data, it may be crucial to consider the sign of a relation. Consider the following example:

> You are a manager in an organization and have to assess whether to promote a certain candidate $t$ that you do not know first-hand. Besides other data and means of assessment, you do know two employees $s_1$ and $s_2$ who work in the same unit as $t$. For instance, you think that $s_1$ is someone who cooperates well, while $s_2$ is rather not cooperative (say $s_1$ makes very constructive contributions to work teams, while $s_2$'s contributions are rather impeding progress). Moreover, you know the relationship among employees, as illustrated in Figure 1. Now, the fact that candidate $t$ holds a positive relation to the cooperative employee $s_1$ might yield a positive signal about the cooperativeness of $t$. Moreover, this belief could be reinforced by $t$'s negative relation to employee $j$, who is positively related to the non-cooperative employee $s_2$. Finally, the positive view might be somewhat challenged by the fact that candidate $t$ holds a positive relation with employee $i$, who is, in turn, negatively related to the cooperative employee $s_1$. Overall, the way candidate $t$ is related to $s_1$ and $s_2$ might be informative about his own cooperativeness, here yielding rather positive signals. (This assessment would lead to quite different conclusions if, for instance, employee $j$ were to judge.)

[1]For instance, Harrigan et al. (2020) provide a long list of examples for negative ties and argue that negative ties and signed networks are under-studied and have only recently received more attention.

More generally, for agents (nodes) whose qualities the observer does not know first hand, she can still try to infer them from other nodes and the link signs between them of the network. This is particularly important for finding hidden qualities of agents like trustworthiness, endurance, cooperativeness, honesty, compliance to norms, etc. There are several applications with a similar characteristic: Networks among employees (as in the example); inter-firm networks (e.g. which other small businesses to trust); criminal networks (who collaborates with the mafia and who acts legal); friendship networks (e.g. is someone a suitable roommate); two political camps (e.g. tell revolutionists from governmentalists); alliances between countries (see e.g. Harary, 1961); networks of artists (e.g. whether a certain book/film/song probably fits one's taste). The fundamental assumption is that link signs are related to the node type. The classic theory of structural balance provides a foundation for this assumption (Cartwright and Harary, 1956).[2] Harary et al. (1953) show that a signed network is balanced if and only if it has a two-camp structure. The two-camp structure, which Harary calls "polarization" means that nodes can be organized in two camps such that there are only positive links within each camp and negative links across.

In this paper, we provide a model of learning from link signs. An observer is given a signed network and information about the types of some source nodes. Assuming that the sign of a link is related to node type, based on patterns of structural balance, the observer tries to learn about the type of the target node. We denote by $r$ (for link reliability) the probability that a given link between two nodes of the same type is positive and that given a link across different types is negative. Similarly, $q$ (for signal quality) denotes the probability that the signal about the type of a source node is correct. We explicitly derive the globally optimal inference, which we call the *Bayesian rule*. This rule provides a natural benchmark for how much there is to learn. Unfortunately, it is complicated to apply, as it involves considering $2^N$ states, namely all combinations of initial node types. We propose a much simpler heuristic that is based on the shortest paths between source nodes and target nodes. This *shortest path rule* considers just one shortest path between a source node and a target node and uses the information of the link signs along this path to judge the target node. For multiple source nodes it aggregates the information as if the paths were non-overlapping.

We assess the accuracy of this shortest path rule in comparison to the Bayesian benchmark and to another heuristic, the *random neighbor rule*.[3] We first derive explicitly how

[2]Basically, if a friend of a friend is my friend, this 3-cycle is consistent with balance, but if a friend of a friend is my enemy, this 3-cycle is not consistent with balance. More generally and more formally, a signed network is called *balanced* if for every cycle the number of negative links is even (Cartwright and Harary, 1956).

[3]The random neighbor rule was introduced in Medo et al. (2020) to model a cognitively-simple opinion

expected accuracy of the shortest path rule depends on link reliability $r$, on signal quality, $q$, and on the distance between source and target node (Claim 3). As the main theoretical result, we show that the three rules can be ordered in terms of expected accuracy: the random neighbor rule is weakly dominated by the shortest path rule, which is weakly dominated by the Bayesian rule (Proposition 1). Only in a small class of networks (that includes trees), i.e. single source node with a unique path to a target node, the three rules lead to the same judgment, which means that the shortest path rule and the random neighbor rule are also optimal (Proposition 2). In a larger class of networks (that contains the small class above) that can be described as a star-like structure with the target node at the center, i.e. each source node has a unique path to target node that is non-overlapping with the other paths, the shortest path rule is still optimal, but the random neighbor rule is in general not (Claim 1). Without restriction to these classes, i.e. in arbitrary connected networks, the shortest path rule is in general not optimal. The reason is that it treats information of the source nodes *as if* there was a star-like structure around the target node, as we illustrate with counter-examples. This behavioral mistake is similar to known imperfection in belief formation, such as correlation neglect (Enke and Zimmermann, 2019).

We use simulations to quantify the differences in accuracy between the three rules. The simulations assess the accuracy for different levels of link sign reliability ($r$), for different quality of initial signals about source nodes ($q$), for different network structures, and for different sizes of the network. The simulation results show that differences can be substantial. The shortest path rule is, in particular, better in handling multiple source nodes than the random neighbor rule. The intuition is as follows: When there are more source nodes, both rules benefit from shorter paths to the target node, but only the shortest path rule aggregates information of multiple paths. Therefore, when increasing the network size, i.e. the number of nodes in the network, and proportionally increasing the number of source nodes, the accuracy of the shortest path rule can even increase. We show that the crucial network statistic is the average distance. For large networks, the heuristics fail to produce accurate results, except for the shortest path rule if the number of source nodes grows with network size and link reliability $r$ is large enough.

Our work is most strongly related to the work of Medo et al. (2020). They introduce a model, that translates into ours, and show in particular that the random neighbor rule fails for large networks. We contribute to this very young literature by introducing another heuristic, the shortest path rule, and comparing both the shortest path rule and the random neighbor rule to the theoretically optimal solution, the Bayesian rule.

formation process where each new opinion is decided strictly locally based on an already formed opinion on a neighboring node.

Our work is also related to the literature on signed network mining. While node ranking and community detection are tasks that are also studied for signed networks, this is rarely the case for the task of node classification (Tang et al., 2016b). Tang et al. (2016a) provide a first framework for the classification of signed networks and have to resort to algorithms that mostly ignore the negative links. In some sense, we also classify nodes as the observer judges their type. We do not only provide a new algorithm that classifies a target node into two types in signed networks, but also solve for the optimal solution within the setting of our model. While node ranking in signed networks is a different objective than classification, its algorithms can also be strongly related to our work. For instance, in Ortega et al. (2012) prior information on which nodes to trust and which nodes to distrust is combined with a PageRank-like propagation mechanism over a signed network. There are clear differences between our work and this literature: Instead of using a data-driven, e.g. machine learning approach, we study the theoretically optimal solution and compare it to simpler heuristics. Our exercise sheds light on why classification of node types works in certain settings and fails in others.

Research on signed networks has always had an emphasis on structural balance, which is still true in recent contributions. Some recent studies measure the extent of structural balance in real social networks and find mixed evidence (Leskovec et al., 2010; Lerner, 2016; Kirkley et al., 2019). Other recent studies address the dynamics of structural balance and hence the formation of signed networks (Derr et al., 2018; Zhao et al., 2017; Rabbani et al., 2019). These works relate to our paper in that more or less structurally balanced societies are the basis of our model. Interestingly, Lerner (2016) argues that by counting the occurrence of patterns that are consistent or inconsistent with structural balance, the basic problem is that the presence of links and the link signs are mingled. He proposes that "estimating the *conditional* [emphasis in the original source] probability of a tie having a particular sign, given that there is a tie, is a more appropriate operationalization and has a clearer interpretation." In our model, we disentangle the two by assuming that only link signs depend on node types, while the presence or absence of a link does not.

Finally, within the large literature about social learning and opinion dynamics in social networks, a few papers consider signed networks (see Shi et al., 2019 for a review and, e.g., Li, 2021 for a recent piece of work). These models generalize the classic DeGroot model to include negative links and study dynamics of opinions on a given topic when agents interact repeatedly. Our work differs fundamentally as we consider an observer who forms beliefs about a social network, instead of forming a belief by interacting within a social network. An interesting analogy to this literature is that simple rules of learning that ignore some complex aspects of the social network can be compared to the benchmark of optimal Bayesian learning.

Our results are important because they show when the two simple heuristics work and when they fail. Beyond assessing the performance of different rules, they compare the heuristics to the theoretically optimal rule and hence show why the heuristics sometimes work and sometimes fail.

# 2 Model

We first define the set-up, then introduce the inference rules, and finally turn to the outcome measures.

## 2.1 Set-Up

We construct a signed network as follows. Let $(V, E)$ be a simple graph of $N := |V|$ vertices (or nodes) and with the set of edges (or links) $E$. The graph is assumed to be connected.[4] For each node $i \in V$, the type $\theta_i \in \{-1, +1\}$ is drawn independently at random with equal probability such that $P[\theta_i = +1] = P[\theta_i = -1] = \frac{1}{2}$. A signed network can be represented by a symmetric matrix $\boldsymbol{R}$ with $R_{ij} \in \{-1, 0, +1\}$, where $R_{ij} = 0$ if there is no link between $i$ and $j$. For each link $(i, j) \in E$, there is either a positive sign $R_{ij} = +1$ or a negative sign $R_{ij} = -1$. The link signs are drawn independently and conditionally on node types as follows: between nodes of the same type (different types) the probability of a positive (negative) link sign is $r \in (0.5, 1]$, i.e. for any link $(i, j)$, $P[R_{ij}\theta_i\theta_j = +1] = r$.

Some nodes $S \subset V$ are pre-determined as source nodes. For each source node $i \in S$, the observer receives an independent signal $\sigma_i \in \{-1, +1\}$ about its type. Each signal correctly indicates the node type with probability $q \in (0.5, 1]$, i.e. $P[\sigma_i\theta_i = +1] = q$. We collect all initial signals in a vector $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_N)$, where $\sigma_i \in \{-1, +1\}$ for $i \in S$, and $\sigma_i = 0$ for $i \notin S$. Parameters $r$ and $q$ are referred to as *link reliability* and source *signal quality*, respectively. We assumed $r, q > 0.5$ because the signed network contained no information when either $r$ or $q$ were 0.5.

Given a signed network $\boldsymbol{R}$ and signals $\boldsymbol{\sigma}$, an observer wants to judge the quality of some target node $t \in (V \setminus S)$. To this end, the observer might apply one of the following three rules.

[4]For network terms that are standard in graph theory and network science (such as connected graph, path, distance), we refer the reader to textbooks, e.g. Diestel (2017) and Jackson (2010).

## 2.2 Three Inference Rules

**Optimal Benchmark: The Bayesian Rule (Bayes).** The Bayesian rule properly takes into account all available information and forms the corresponding posterior belief. Each state of the world is one combination of node types $\boldsymbol{\theta} = (\theta_1, ..., \theta_N) \in \Theta = \{-1, +1\}^N$. As $\theta_i \in \{-1, +1\}$ is binary, there are $2^N$ states of the world. Since node types are independently drawn with equal probability, the prior probability of each state $\boldsymbol{\theta}$ is the same, $P[\boldsymbol{\theta}] = (\frac{1}{2})^N$. The conditional probability that, given a true state $\boldsymbol{\theta}$, we observe signals $\boldsymbol{\sigma}$ and link signs $\boldsymbol{R}$ is

$$P[\boldsymbol{\sigma}, \boldsymbol{R}|\boldsymbol{\theta}] = q^{z_1(\boldsymbol{\theta})}(1-q)^{z_2(\boldsymbol{\theta})} r^{z_3(\boldsymbol{\theta})}(1-r)^{z_4(\boldsymbol{\theta})},$$

where $z_1(\boldsymbol{\theta}) := \#\{i \in S | \sigma_i \theta_i = +1\}$ is the number of source nodes for which the signal is correct; $z_2(\boldsymbol{\theta}) := \#\{i \in S | \sigma_i \theta_i = -1\}$ is the number of source nodes for which the signal is wrong; $z_3(\boldsymbol{\theta}) := \#\{(i,j) \in E | R_{ij} \theta_i \theta_j = +1\}$ is the number of links for which the link sign is according to the node types; and $z_4(\boldsymbol{\theta}) := \#\{(i,j) \in E | R_{ij} \theta_i \theta_j = -1\}$ is the number of links for which the link sign is not according to the node types. Note that $z_1(\boldsymbol{\theta}) + z_2(\boldsymbol{\theta}) = N_S$, where $N_S := |S|$ is the number of source nodes; and $z_3(\boldsymbol{\theta}) + z_4(\boldsymbol{\theta}) = |E|$.

Bayes' rule gives us the conditional probability that, given observations $\boldsymbol{\sigma}, \boldsymbol{R}$, the true state of the world is $\boldsymbol{\theta}$:

$$P[\boldsymbol{\theta}|\boldsymbol{\sigma}, \boldsymbol{R}] = \frac{P[\boldsymbol{\sigma}, \boldsymbol{R}|\boldsymbol{\theta}] \cdot P[\boldsymbol{\theta}]}{P[\boldsymbol{\sigma}, \boldsymbol{R}]} = \frac{q^{z_1(\boldsymbol{\theta})}(1-q)^{z_2(\boldsymbol{\theta})} r^{z_3(\boldsymbol{\theta})}(1-r)^{z_4(\boldsymbol{\theta})} \cdot (\frac{1}{2})^N}{\sum_{\boldsymbol{\theta}' \in \Theta} q^{z_1(\boldsymbol{\theta}')}(1-q)^{z_2(\boldsymbol{\theta}')} r^{z_3(\boldsymbol{\theta}')}(1-r)^{z_4(\boldsymbol{\theta}')} \cdot (\frac{1}{2})^N}. \quad (1)$$

While the numerator uses the likelihood of a given state $\boldsymbol{\theta}$, the denominator uses all possible states, $\boldsymbol{\theta}' \in \Theta$, to generate the probability of the actual observation $\boldsymbol{\sigma}, \boldsymbol{R}$. Note that the terms $(\frac{1}{2})^N$ cancel out.

Finally, the probability that target node $t$ is of the positive type is the probability that we are in a state of the world with $\theta_t = +1$. Let $\Theta(t)$ be all $2^{N-1}$ states with $\theta_t = +1$. Then the posterior belief that target node $t$ is of positive type, given observations $\boldsymbol{\sigma}, \boldsymbol{R}$, is

$$b^{Bayes}(t) := \sum_{\theta \in \Theta(t)} P[\boldsymbol{\theta}|\boldsymbol{\sigma}, \boldsymbol{R}], \quad (2)$$

where $P[\boldsymbol{\theta}|\boldsymbol{\sigma}, \boldsymbol{R}]$ is given by Eq. (1).[5]

For an observer (who knows which model has produced $\boldsymbol{\sigma}, \boldsymbol{R}$ and knows the values of $q$ and $r$), the Bayesian rule is the optimal way to process the available information. The downside is that its application involves $2^N$ states which makes its direct use prohibitively

[5]From the posterior belief, we can also derive the expected node type: $E[\theta_t|\boldsymbol{\sigma}, \boldsymbol{R}] = P[\theta_t = +1|\boldsymbol{\sigma}, \boldsymbol{R}] \times 1 + (1 - P[\theta_t = +1|\boldsymbol{\sigma}, \boldsymbol{R}]) \times (-1)$.

computationally demanding. We apply the Bayesian rule theoretically and restrict simulations with the Bayesian rule to $N$ up to 20.[6]

**A New Heuristic: The Shortest Path Rule (ShPath).** Instead of going through all complexities that the Bayesian rule necessitates, the observer might simply consider how the target node is connected to a given source node by a shortest path. We propose a new rule based on this idea. The *shortest path rule* is defined as follows:

1. Select one source node $s \in S$. Identify all shortest paths from the source node $s$ to the target node $t$ and choose one at random. Label nodes on the path "$s \to t$" by $n_0, n_1, \ldots, n_L$ where $n_0 := s$, $n_L := t$ and $L$ is the path length.

2. Compute the *path orientation* produced by both the signal and link signs along the path
$$o_{s\to t} := \sigma_s \prod_{m=0}^{L-1} R_{n_m, n_{m+1}}. \tag{3}$$

3. Compute the probability that the shortest path has an even number of errors (i.e., links whose sign does not match the node types)[7]
$$\pi_{s\to t} := r^L + \binom{L}{2} r^{L-2}(1-r)^2 + \cdots = \frac{1}{2}\Big[1 + (2r-1)^L\Big]. \tag{4}$$

4. Determine the belief derived from this path as
$$b_{s\to t} := \begin{cases} q\pi_{s\to t} + (1-q)(1-\pi_{s\to t}) & \text{if } o_{s\to t} = +1, \\ 1 - q\pi_{s\to t} - (1-q)(1-\pi_{s\to t}) & \text{if } o_{s\to t} = -1. \end{cases} \tag{5}$$
We can interpret $b_{s\to t}$ as the subjective probability that node $t$ is of positive type, as it can be derived from the path from $s$ to $t$. Consider, for instance, the first case, where the path's orientation is positive, i.e. where $o_{s\to t} = +1$. There the former term is the probability that the source node signal is correct *and* the number of erroneous links between $s$ and $t$ is even which together imply that $o_{s\to t} = \theta_t$. The latter term in Eq. (5) is the probability that the source node signal is incorrect *and* the number of erroneous links between $s$ and $t$ is odd, which again implies that $o_{s\to t} = \theta_t$. Hence, both terms together yield the probability that the path orientation, which is

[6] Approximate schemes based on Markov Chain Monte Carlo (Brooks et al., 2011) could be developed to be able to treat larger values of $N$.

[7] The closed-form expression is obtained by replacing the sum over even $j$ with $[r + (1-r)]^L/2 + [r - (1-r)]^L/2$.

positive, coincides with the type of the target node $t$. Analogously, the second case yields the probability that the path orientation, which is negative, does not coincide with the type of the target node.

5. Repeat steps 1–4 for all source nodes $S$. Compute the belief (i.e. subjective probability) that target node $t$ is of positive type as

$$b^{ShPath}(t) := \frac{\prod_{s\in S} b_{s\to t}}{\prod_{s\in S} b_{s\to t} + \prod_{s\in S}(1 - b_{s\to t})}, \tag{6}$$

which aggregates beliefs $b_{s\to t}$ from each path, as if they were containing independent information. For one source node ($N_S = 1$), we have $b^{ShPath}(t) := b_{s\to t}$ as given by Equation (5).

The application of this rule is actually quite simple. Suppose first that there is only one source node, i.e. $S = \{s\}$ and $N_S = 1$. Then the *path orientation* $o_{s\to t}$, which is derived in step 2, already indicates the best guess about the target node's type: A positive (negative) path orientation leads to the best guess that the target node is rather of the positive (negative) type. Steps 3 and 4 attach a probability to this guess. Equation (5) (in step 4) accounts for the probability $q$ that the source node's signal is accurate. Finally, in step 5 we obtain belief $b^{ShPath}(t) := b_{s\to t}$ when there is only one source node. For multiple source nodes the procedure is repeated for each source node. Equation (6) in step 5 then aggregates information from all source nodes *as if* the shortest paths were independent.[8]

The shortest path rule is motivated by an observer who is less sophisticated than the Bayesian observer. In particular, this observer ignores information from other shortest paths (if multiple shortest paths exist between $s$ and $t$), from other paths that are not shortest, and the fact that the multiple shortest paths might be overlapping.[9]

**A Known Heuristic: The Random Neighbor Rule (RNeighbor).** In a recent paper, Medo et al. (2020) introduced a closely-related framework and studied the *random*

[8]This formula weighs the likelihood that all beliefs are correct against the likelihood that all are correct *or* all are wrong. For example, if one source node would lead to a belief of 0.8 and the other source node to a belief of 0.6, then the belief from both source nodes is $(0.8*0.6)/[0.8*0.6+(1-0.8)*(1-0.6)] \approx 0.857$. Several simple examples that illustrate the application of all three rules are presented in Sections 2.3 and Appendix B.

[9]In spirit, this is similar to imperfections in belief formation and social learning. For instance, there are several biases when people learn about probabilities (e.g. Benjamin, 2019); including correlation neglect, i.e. assuming that signals are independent even if they are correlated (Enke and Zimmermann, 2019). Also non-Bayesian models of social learning (or opinion dynamics), are motivated by naïveté in updating. For instance, (DeMarzo et al., 2003) justify the DeGroot model with "persuasion bias," a tendency to ignore that several pieces of information may be redundant, e.g. stemming from the same source.

*neighbor rule*.[10] In the random neighbor rule, the observer uses signals on the source node types to form opinions $x_i \in \{-1, +1\}$ for all other nodes. The opinion formation process is step-wise:

1. Judge every source node $s \in S$ according to its signal: $x_s = \sigma_s$.

2. Choose node $j$ to judge at random from all nodes that have not been judged yet. If no neighbor of $j$ has been judged yet, repeat the choice.

3. From the judged neighbors of node $j$, choose neighbor $i$ at random. The judgment of node $j$ is then $x_j = x_i R_{ij}$.

4. If target node $t$ has not been judged yet, continue at step 2. If the target node has been judged, then we define the best guess about the target node's type as $g^{RNeighbor}(t) := x_t$.

There is a difference between the random neighbor rule and the two other rules, in terms of the output. The *Bayesian rule* and the *shortest path rule* produce beliefs $b(t)$ that lie in the interval $[0, 1]$ and express the observer's subjective probability that the target node is of the positive type $\theta_t = +1$. The *random neighbor rule* makes a judgement $x_t$ about the target node that is either $+1$ or $-1$ without attaching a level of confidence to it, i.e. $g^{RNeighbor}(t) \in \{-1, 1\}$. To allow for a fair comparison between the three rules, we map each continuous belief $b(t) \in [0, 1]$ into a "best guess" $g(t) \in \{-1, +1\}$. In particular, let $g(t) = +1$ for $b(t) > 0.5$, $g(t) = -1$ for $b(t) < 0.5$. In the knife-edge case $b(t) = 0.5$, the best guess is chosen from $\{-1, +1\}$ at random, with equal probability.

## 2.3 Illustration of How the Rules are Applied

Before we systematically compare the three rules, let us first illustrate how they are applied in two simple examples. Both examples are subgraphs of the signed network among employees illustrated in Figure 1.[11]

**Example 1** (A Triangle)**.** *Focus on the top part of Figure 1, i.e. on the subgraph* $V' = \{s_1, i, t\}$ *with edges* $E' = \{s_1t, s_1i, it\}$*. The source node's signal is* $\sigma_{s_1} = +1$ *and the link signs are* $R_{s_1t} = R_{it} = +1$ *and* $R_{s_1i} = -1$*.*

[10] Our model framework translates to theirs when we set link reliability $r \equiv 1 - \beta$ where $\beta$ is their noise parameter and fix signal quality to be $q \equiv 1$. The other rule studied in (Medo et al., 2020), the Majority rule, is less prominent than the random neighbor rule, so we omit it from our analysis.

[11] More details about these two examples and a solution to the whole network among employees as depicted in Figure 1 is provided in Appendix B.1.

*The Bayesian rule considers all* $2^3 = 8$ *states of the world. For instance, state* $\boldsymbol{\theta} = (\theta_{s_1}, \theta_i, \theta_t) = (+1, +1, +1)$*, in which all nodes are of positive type, would generate the observations* $\boldsymbol{\sigma}, \boldsymbol{R}$*, with probability* $P[\boldsymbol{\sigma}, \boldsymbol{R}|\boldsymbol{\theta}] = qr^2(1-r)$*, as the signal for* $s_1$ *agrees with its type and two out of three links agree with their node types. Bayes' rule gives us the conditional probability that, given observations* $\boldsymbol{\sigma}, \boldsymbol{R}$*, the underlying state is indeed* $\boldsymbol{\theta}$*:* $P[\boldsymbol{\theta}|\boldsymbol{\sigma}, \boldsymbol{R}] = P[\boldsymbol{\sigma}, \boldsymbol{R}|\boldsymbol{\theta}] \cdot P[\boldsymbol{\theta}]/P[\boldsymbol{\sigma}, \boldsymbol{R}]$ *by Equation* (1)*, which is a somewhat tedious expression. With Equation* (2) *we finally receive the Bayesian posterior belief that target node* $t$ *is of positive type, given observations* $\boldsymbol{\sigma}, \boldsymbol{R}$*, and simplify it to*

$$b^{Bayes}(t) = P[\theta_t = +1|\boldsymbol{\sigma}, \boldsymbol{R}] = \frac{(1+q)r^2 + (1-q)(1-r)^2}{3r^2 + (1-r)^2}.$$

*Next, let us apply the shortest path rule, again to the subgraph* $(V', E')$*. The only shortest path from source to target node is obviously* $s_1, t$ *and has length* 1*. The path's orientation is* $o_{s_1 \to t} = \sigma_{s_1} R_{s_1,t} = (+1) \times (+1) = 1$*, which already means that the best guess is that the target node is rather of the positive type. The probability that the shortest path has an even number of errors* $\pi_{s_1 \to t} = r$ *by Equation 4. Hence, the belief derived from this path by Equation* (5) *is* $b_{s_1 \to t} = qr + (1-q)(1-r)$*, which does not change by Equation* (6)*.*

$$b^{ShPath}(t) = qr + (1-q)(1-r).$$

*Finally, let us apply the random neighbor rule. In the first step, the judgment of source node* $s_1$ *is produced by its signal as* $x_{s_1} = \sigma_{s_1} = +1$*. Now, a next node is chosen from* $\{i, t\}$ *at random. If this node is* $t$*, we receive the judgement* $x_t = x_{s_1} R_{s_1 t} = (+1) \times (+1) = +1$ *and we have the best guess* $g^{RNeighbor}(t) = +1$*. Otherwise,* $i$ *is judged next with* $x_i = x_{s_1} R_{s_1 i} = (+1) \times (-1) = -1$*. Then the next node must be* $t$*. It is selected at random whether the judgement on* $t$ *comes from neighbor* $s_1$ *or from neighbor* $i$ *(who are both already judged). In the former case we have* $x_t = x_{s_1} = 1$*; in the latter case we have* $x_t = x_i = -1$*. Overall, we have a* 3/4 *chance to reach* $g^{RNeighbor}(t) = +1$ *(when the judgement about* $t$ *is derived from* $s_1$ *directly) and a* 1/4 *chance to reach* $g^{RNeighbor}(t) = x_i = -1$ *(when the judgement about* $t$ *is derived from* $s_1$ *via* $i$*).*

Computing the beliefs in Example 1 yields $0.5 < b^{Bayes}(t) < b^{ShPath}(t)$. Hence, we can observe that, in this example, the *Bayesian rule* and the *shortest path rule* lead to the same best guess that the target node is of the positive type ($g^{Bayes}(t) = g^{ShPath}(t) = +1$). The random neighbor rule agrees in most, but not all cases. Interestingly, the confidence level of the *Bayesian rule* is lower than that of the *shortest path rule*, i.e. $b^{Bayes}(t) < b^{ShPath}(t)$, since it takes into account that besides the direct shortest path with a positive orientation, there is a second path with a negative orientation. The next example shows how the rules

are applied when there is a single longer path.

**Example 2** (A Line)**.** *Focus now on the lower part of Figure 1, i.e. on the subgraph* $V'' = \{s_2, j, t\}$ *with edges* $E'' = \{s_2 j, jt\}$*. The source node's signal is* $\sigma_{s_2} = -1$ *and link signs are* $R_{s_2 j} = +1$ *and* $R_{jt} = -1$*.*

*The Bayesian rule considers again* $2^3$ *states of the world. It finally yields the belief*

$$b^{Bayes}(t) = [qr^2 + q(1-r)^2 + 2(1-q)r(1-r)]/1 = 4qr^2 - 4qr + 2r - 2r^2 + q.$$

*The shortest path from source to target node is clearly* $s_2, j, t$ *and has length* 2*. The path's orientation is positive as the negative signal for* $s_2$ *and the negative link between* $j$ *and* $t$ *offset each other:* $o_{s_2 \to t} = \sigma_{s_2} R_{s_2 j} R_{jt} = (-1) \times (+1) \times (-1) = 1$*. The probability that the shortest path has an even number of errors is* $\pi_{s_2 \to t} = r^2 + (1-r)^2$ *by Equation 4. Hence, we receive the final belief*

$$b^{ShPath}(t) = q[r^2 + (1-r)^2] + (1-q)[1 - (r^2 + (1-r)^2)] = 4qr^2 - 4qr + 2r - 2r^2 + q.$$

*The random neighbor rule starts with setting* $x_{s_2} = \sigma_{s_2} = -1$*. Then it chooses a node from* $\{j, t\}$ *at random. If the node is* $t$*, it chooses again as the only neighbor of* $t$*,* $j$*, has not been judged yet. If the node is* $j$*, it makes the judgement* $x_j = x_{s_2} R_{s_2 j} = (-1) \times (+1) = -1$*. Finally, it judges node* $t$ *as* $x_t = x_j R_{jt} = (-1) \times (-1) = +1$*. Hence,* $g^{RNeighbor}(t) = +1$*.*

Computing the beliefs in Example 2 yields $0.5 < b^{Bayes}(t) = b^{ShPath}(t)$, while $g^{Neighbor}(t) = +1$. This means first that all rules lead to the same best guess that the target node $t$ is rather of the positive type: $g^{Bayes}(t) = g^{ShPath}(t) = g^{RNeighbor}(t)$. And second that the shortest path rule leads to exactly the same belief as the Bayesian rule and is hence optimal. Both observations are not coincidences, as we prove in Claims 1 and 2 in the next subsection.

# 3 Theoretical Results

## 3.1 Comparison of the Three Rules

The three seemingly very different rules are in fact strongly related to each other.

### 3.1.1 Comparison of Shortest Path Rule with Bayesian Rule

The example of a line graph above (Example 2) already indicates that sometimes the shortest path rule leads to the same belief as the Bayesian rule. The first proposition provides a sufficient condition for this conclusion.

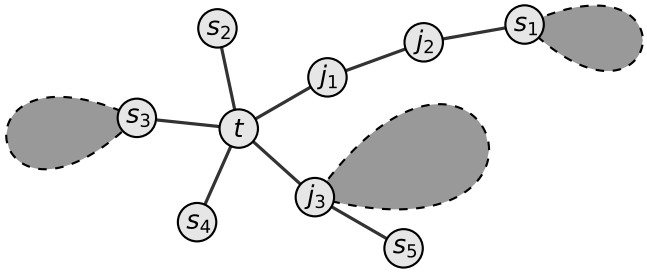


Figure 2: A star-like network with five source nodes $(s_1, \ldots, s_5)$ and subgraphs attached to nodes $s_1$, $s_3$, and $j_3$. As the path between each source node and target node $t$ is unique and does not overlap with other paths, the condition of Claim 1 holds. Hence, the shortest path rule and the Bayesian rule yield identical beliefs about $t$ (for any $\boldsymbol{\sigma}$ and $\boldsymbol{R}$).

**Claim 1** (Equivalence)**.** *Suppose that in* $(V, E)$ *between each source node* $s \in S$ *and the target node* $t$ *there is a* unique *path that is* non-overlapping *with the paths of the other source nodes. Then the Bayesian rule and the shortest path rule are equivalent: for any observation* $\boldsymbol{\sigma}, \boldsymbol{R}$*,*

$$b^{Bayes}(t) = b^{ShPath}(t).$$

The condition of the result (*unique* path that is *non-overlapping*) is equivalent to the following: Between every pair of source nodes, there is one single path and the target node $t$ lies on this path. Hence, these networks have a star-like structure, where target node $t$ is the center and each source node $s$ is on a unique ray (of arbitrary length), plus potentially subgraphs that are "pockets", as illustrated in Figure 2.

Conversely, if these conditions (*unique* path that is *non-overlapping*) are not met, the shortest path rule may be suboptimal. We demonstrate this by three examples which are illustrated in Figure 3, and solved in Appendix B. For concreteness, we suppose in all examples that the source node signals are positive. In Example B.1, illustrated in Panel (A) of Figure 3, there is a unique shortest path between source node and target node, but there are additional (non-shortest) paths. The shortest path rule only uses the direct path, which has a positive orientation. However, the Bayesian rule uses all information including the two paths of length 2, which are both negatively oriented. Hence for $r$ not too small, the best guess of the Bayesian rule is negative, while the shortest path rule comes to the opposite conclusion.

In Example B.2, illustrated in Panel (B) of Figure 3, there are multiple shortest paths between source node and target node. The Bayesian rule always leads to a belief $b^{Bayes}(t) > 0.5$ and corresponding best guess of $+1$, as two out of three paths are positively oriented. The shortest path rule selects one of the three shortest paths at random. Hence,

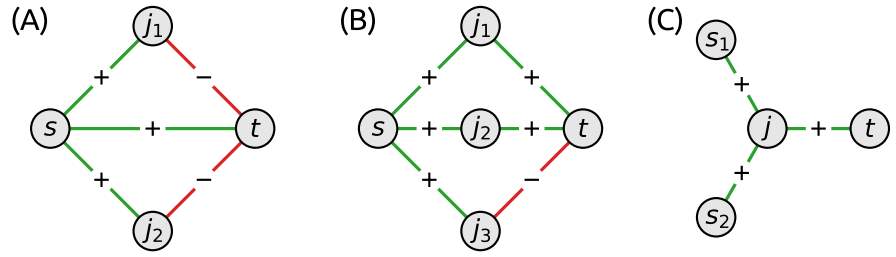


Figure 3: Illustration of signed networks in Examples B.1-B.3. These networks do not satisfy the condition of Claim 1 such that the shortest path rule leads to suboptimal beliefs.

with probability 2/3 it selects a positively oriented path and induces a best guess of $+1$, but with probability 1/3 it induces a best guess of $-1$.

Finally, in Example B.3, illustrated in Panel (C) of Figure 3, there are two source nodes whose shortest paths to the target node overlap. For $r, q \in (0.5, 1)$, we receive $b^{ShPath}(t) > b^{Bayes}(t)$. The suboptimality of the shortest path heuristic in this example enters in step 5 of the procedure when multiple paths are aggregated. The shortest path rule aggregates information of both paths as if they were independent, ignoring that they are correlated by having link $jt$ in common. This mistake is very similar in spirit to persuasion bias in the DeGroot model (DeMarzo et al., 2003) and to correlation neglect (Enke and Zimmermann, 2019). We observe that ignoring this overlap leads to overconfidence, as the subjective probability is higher than the objective probability: $b^{ShPath}(t) > b^{Bayes}(t) > 0.5$.[12]

As we can see in these three examples, the reason for potential suboptimality of the shortest path rule is that it aggregates information from the signed network *as if* all paths from any source node to the target node were unique and independent.

### 3.1.2 Comparison of Shortest Path Rule with Random Neighbor Rule

Recall that the *random neighbor rule* leads to a best guess $g^{Neighbor}(t)$ about the target node's type that can be compared to the corresponding best guess that is induced by the shortest path rule $g^{ShPath}(t)$. The line graph discussed as Example 2 already indicates that the random neighbor rule can sometimes lead to the same best guess as the shortest path rule. The next proposition provides a sufficient condition for this conclusion.

**Claim 2** (Same Best Guess)**.** *Suppose that in $(V, E)$ there is a unique path between the only source node $s$ and the target node $t$. Then for any observation $\boldsymbol{\sigma}, \boldsymbol{R}$, we have*

[12] In larger examples (that can be requested from the authors) it may also lead to different best guesses.

$g^{ShPath}(t) = g^{RNeighbor}(t)$.

For the interpretation of Claim 2 it is helpful to re-interpret the random neighbor rule in terms of paths. When we apply the random neighbor rule using source node $s$, the final judgment of node $t$ is ultimately based on a random path between the source node and the target node. This random path's orientation $o_{s\to t}$ determines the best guess, similarly to the shortest path rule, where the orientation of the shortest path determines the best guess. Hence, when there is a unique path between $s$ and $t$, then this path is also the shortest path and the two rules lead to the same best guess.[13]

Conversely, when there are multiple paths between the single source node $s$ and the target node $t$; or when there is more than one source node, the random neighbor rule does not always lead to the same best guess as the shortest path rule. For instance, in Example 1, illustrated in the upper part of Figure 1, there are multiple paths and the random neighbor rule may lead to a best guess of $-1$, while the shortest path rule leads to $+1$. In Example B.3, illustrated in panel B of Figure 3, there are multiple paths of the same length. It might happen that we have a negative signal for source node $s_1$ and a positive signal for $s_2$ and that the random neighbor rule bases the judgement on the path from $s_1$, while the shortest path rule randomly selects the other shortest path.[14]

## 3.2 Comparison of Accuracy

We now theoretically assess to which extent the three rules deliver a correct judgment.

**Definition 1** (Accuracy)**.** *Let $g(t)$ be the observer's best guess of type of node $t$ obtained using a certain rule, while true node types are $\boldsymbol{\theta}$.*[15] *The rule's accuracy for target node $t$ is defined to be* 1 *for a correct judgement and* 0 *for a wrong judgement:*

$$A(t) = \begin{cases} 1 & \text{if } g(t)\theta_t = +1, \\ 0 & \text{if } g(t)\theta_t = -1. \end{cases}$$

This definition allows us to assess the obtained opinion for any given realization of node types, source signals, and link signs. However, such a realization is not necessar-

[13]Moreover, since the conditions of Claim 2 are stronger than those of Claim 1, it actually follows that all three rules lead to the same best guess under the conditions of Claim 2, i.e. in the special case that there is only one source node and a unique path between source node and target node. This insight will lead to Proposition 2 below.

[14]However, in this network the two rules would lead to the same quality of judgment in expectations – a topic that we address in the next subsection.

[15]Recall that for any belief $b(t) > 0.5$ the best guess is $+1$, for any belief $b(t) < 0.5$ the best guess is $-1$, and for $b(t) = 0.5$ the best guess is chosen from $\{-1, +1\}$ with equal probability.

ily representative. To measure the performance of an inference rule, we thus assess its expected accuracy.

**Definition 2** (Expected Accuracy)**.** *Fix a network $(V, E)$ with source nodes $S$ and target node $t$ and also fix the model parameters $q$ and $r$. A rule's* expected accuracy, *$E[A]$, is the expected value of accuracy over all different realizations (states $\boldsymbol{\theta}$, signals $\boldsymbol{\sigma}$ and link signs $\boldsymbol{R}$) and all different judgment outcomes.*[16]

Expected accuracy can be interpreted as the ex ante probability that the final judgement of the target node will be correct. If our belief $b(t)$ correctly reflects the probability that target node $t$ is of the positive type, then $E[A(t)] = b(t)$ if $b(t) \geq 0.5$ and $E[A(t)] = 1 - b(t)$ if $b(t) < 0.5$.

While expected accuracy $E[A]$ can in principle lie anywhere in the range $[0, 1]$, already with random guessing of target node types, we achieve an expected accuracy of $E[A] = 0.5$.[17] Hence, 0.5 is a lower bound on the performance of a sensible rule. At the other extreme, expected accuracy of any rule is bounded from above by the initial information given through the source nodes. For the case of one source node, $N_S = 1$, we have in particular that $E[A] \leq q$ for any rule; the equality is achieved when $r = 1$. The generalization of this upper bound to multiple source nodes is straightforward: with $N_S$ source nodes, the highest possible expected accuracy is achieved when $r = 1$ and it equals the probability that the majority of $N_S$ signals is correct plus one half the probability that there are equally many correct and false signals.

The newly defined expected accuracy allows us to understand how is the performance of the shortest path rule affected by positions of source and target nodes in the network.

**Claim 3** (Shortest Path Accuracy)**.** *Let $N_S = 1$ and suppose that the shortest path between source node $s$ and target node $t$ has length $L$. The expected accuracy of the shortest path rule is then*

$$E[A^{ShPath}] = \tfrac{1}{2} + \left(q - \tfrac{1}{2}\right)(2r - 1)^L. \tag{7}$$

As $o_{s \to t}$ is the best guess of $\theta_t$ for the shortest path rule, the expected accuracy introduced by Definition Eq. (2) is thus equal to $P[o_{s \to t} = \theta_t]$. This lemma then follows by combining with Eqs. (4) and (5). The obtained expected accuracy depends on the

[16] The probabilistic element in the shortest path rule is which shortest path to use if there are multiple. The probabilistic element in the random neighbor rule is the order of judgment. The Bayesian rule is deterministic.

[17] In the work of Medo et al. (2020), a closely-related metric, opinion consistency, is used to assess the opinions formed on several nodes. Since a simple linear relation exists between accuracy $A(t)$ and node $t$'s contribution to opinion consistency, $C(t) = 2A(t) - 1$, the results are qualitatively the same for both of them.

model parameters, $q$ and $r$, and the shortest path length, $L$. When $q = 0.5$ (the source signals are not informative) or $r = 0.5$ (the links are not informative), $E[A^{ShPath}] = 0.5$ for any shortest path length $L$. When $r = 1$ (fully reliable links), $E[A^{ShPath}] = q$ for any $L$. Finally, when $q > 0.5$ and $0.5 < r < 1$ (the non-trivial case), $E[A^{ShPath}]$ decreases as $L$ grows.

Note that Eq. (7) can be used to compute the expected accuracy of the random neighbor rule. Using a path whose length $L$ is greater than the shortest path length directly results in a lower expected accuracy.

We are now in the position to state the main result. It establishes how the three rules are ordered in terms of expected accuracy.

**Proposition 1** (Ordering)**.** *For every given network $(V, E)$, set of source nodes $S$ and target node $t$, the accuracy expectations of the three rules are ordered as*

$$E[A^{Bayes}] \geq E[A^{ShPath}] \geq E[A^{RNeighbor}].$$

The first inequality is a simple implication of the fact that the Bayesian rule computes the correct posterior belief for any realization of source node signals $\boldsymbol{\sigma}$, and link signs $\boldsymbol{R}$. When the observer uses this rule, then her belief (i.e. subjective probability) equals exactly the objective probability that the target node $t$ is of positive type. Hence, the expected value of accuracy after realization of observations $\boldsymbol{\sigma}, \boldsymbol{R}$ (we might call this "ad interim expected accuracy") is maximal. Since this holds for every realization of observations, also the ex ante expected value of accuracy, that is expected accuracy $E[A^{Bayes}]$, is maximal.

The comparison between shortest path rule and random neighbor rule is based on the previously mentioned observation that the random neighbor rule effectively uses a random path between some source node $s$ and target node $t$. If this path is a shortest path, then the shortest path might use the same path or another path of the same length. Both would lead to the same expected accuracy when there are no other source nodes. If the path used by the random neighbor rule is not a shortest path, then the shortest path rule's expected accuracy is weakly higher due to Eq. (7). If there are multiple source nodes, the shortest path rule aggregates information obtained using shortest path from all source nodes. For the random neighbor rule, the judgment of every target node can be traced back to a path from a single source node and the information provided by other source nodes is neglected. This further lowers the expected accuracy of the random neighbor rule which is thus always weakly dominated by the shortest path rule in terms of expected accuracy.

Taken together, the Bayesian rule weakly dominates the shortest path rule, which in turn weakly dominates the random neighbor rule. To be clear, this does not mean that

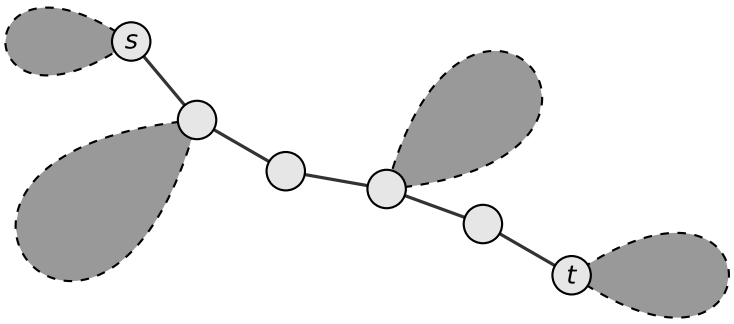


Figure 4: Illustration of Proposition 2. When there is a unique shortest path between source node $s$ and target node $t$, the three rules induce the same accuracy.

there are no instances, where a dominated rule is accurate and a undominated rule is not. For instance, we can have, e.g., $A^{RNeighbor} = 1$ while $A^{ShPath} = 0$ in some realization, but in expectations the ordering $E[A^{ShPath}] \geq E[A^{RNeighbor}]$ holds for any network.

Since Proposition 1 only establishes a weak dominance relation between the three rules, the consequent question is when the three rules do not differ in accuracy. The next result identifies a condition for all three rules to be equally accurate.

**Proposition 2** (Unique Path)**.** *Let $N_S = 1$. Suppose that in network $(V, E)$ there is a unique path between source node $s$ and the target node $t$. Then all three rules (Bayes, ShPath, RNeighbor) lead to the same expected accuracy:*

$$E[A^{Bayes}] = E[A^{ShPath}] = E[A^{RNeighbor}].$$

This proposition follows from the Claims 1 and 2, which establish when the shortest path rule leads to the same belief as the Bayesian rule and when it leads to the same best guess as the random neighbor rule. Indeed, the condition of Proposition 2, i.e., there is a unique path between the only source node and the target node, is equal to the condition of Claim 2, which is stronger than the condition of Claim 1. Hence, all three rules must lead to the same best guess and hence to equal accuracy under this stronger condition. As the three accuracy values are equal for each realization of states, signals, and link signs, they are also equal on average.

The intuition for this result is illustrated by Figure 4. First, the existence of a unique path from $s$ to $t$ implies that the network must look as shown in this figure, where subgraphs are attached to individual nodes of the path. Second, these subgraphs do not affect the inference of the three considered rules. Finally, all three rules lead to the same best guess when only the path from $s$ to $t$ is considered.

Proposition 2 shows that the three rules lead to the same accuracy when there is a unique path between the only source node $s$ and target node $t$. For instance, it implies that this holds for the class of tree networks, as trees are characterized by a unique path between any two nodes. Conversely, it turns out that if there are multiple paths or if there are multiple source nodes, then the three rules' expected accuracy does not coincide, apart from special cases. This, opposite direction of the statement – the three rules lead to the same expected accuracy *only if* there is a unique path between the single source node and the target node – is intuitive when considering again that the random neighbor rule effectively uses one random path from source node to target node. Any additional path and any additional source node provides additional information that gives the Bayesian rule an edge over the random neighbor rule. The shortest path partially uses this additional information. Its accuracy benefits from any additional source node and also from additional paths if they are shorter. The Bayesian rule has an edge over the shortest path rule since it uses *all* paths between a given source node $s$ and the target node $t$ and since it properly combines the information from multiple source nodes (taking into account overlapping paths). However, this opposite direction of the statement does not hold in general. For instance, if $r = 0.5$ or $q = 0.5$ (observations yield no useful information), the expected accuracy of all rules is 0.5, independent of the network structure. And there are examples with multiple source nodes, in which the three rules coincide even for non-trivial values of $r$ and $q$.[18]

In this section we have theoretically established the order of the three rules according to expected accuracy. To quantify these differences between the three rules, we will now turn to simulation results.

# 4 Simulation Results

We first define the simulation procedure.

## 4.1 Simulation Procedure

For each of the three rules (*Bayesian rule*, *shortest path rule*, *random neighbor rule*), the simulations proceed as follows:

1. Generate a synthetic random network of a given class with $N$ nodes and mean degree $z$. If it is not connected, generate a new network. The classes of random networks

[18]For instance, in panel C of Figure 3, the two paths either have the same orientation such that the best guess coincides for all three rules, or the paths have opposite orientations such that the probability of an accurate judgment is one half for all three rules.

that we use are: Erdös-Rényi random networks (ER), Regular random Graphs (RG), Barabási-Albert scale-free networks (BA), and Watts-Strogatz small-world networks (WS).[19]

2. Choose $N_S$ nodes at random to serve as the source nodes.

3. Draw node types, link signs, and source node signals according to the model introduced in Section 2.1.

4. For each node that is not a source node, designate it as target node, apply a given rule (*Bayes*, *ShPath*, or *RNeighbor*) according to Section 2.2, and measure its accuracy $A$.

5. Repeat the above steps $1,000$ times. From the obtained accuracy values, compute mean accuracy, $\overline{A}$, and the standard error of the mean (SEM), $\sigma_{\overline{A}}$. (Figures reporting simulation results depict the mean with error bars that are twice the SEM.)

We apply this procedure for various model parameters ($q$, $r$), network topologies, and network sizes $N$.

## 4.2 Varying Signal Quality and Link Reliability

From the theoretical results we know the ordering of the three rules: $E[A^{Bayes}] \geq E[A^{ShPath}] \geq E[A^{RNeighbor}]$ (Proposition 1) and that they coincide in special cases (Proposition 2). Let us now assess how large the difference between the three rules are. Figure 5 illustrates how accuracy depends on the applied rule for different levels of signal quality $q$ and link reliability $r$. The illustrated simulations use Erdös-Rényi random networks of $N = 20$ nodes with mean degree $z = 5$. The left panels show results for $N_S = 1$ source node, the right panels for $N_S = 5$ source nodes.

Clearly, accuracy of all rules is weakly increasing in signal quality and weakly increasing in link reliability $r$. In the extreme case of $r = 0.5$, we have accuracy of random guessing since the information about the source node(s) cannot be used to judge other nodes when links are independent of type. The other extreme case of $r = 1$, leads to relatively high accuracy and will be further discussed below. Most importantly, Figure 5 illustrates that for intermediate values of link reliability $r$, the rules significantly differ in accuracy. For instance, in the top left panel at $r = 0.8$, accuracy of the Bayesian rule (around 80%) is substantially higher than accuracy of the shortest path rule (around 70%), which in turn is substantially higher than accuracy of the random neighbor rule

[19]Further network generation details are provided in Appendix C.1.

(slightly above 60%). Considering the other depicted settings of signal quality $q$ and number of source nodes $N_S$, there are again levels of link reliability $r$ with significant differences between the three rules. Hence, we receive the following first result:

**Result 1.** *Differences in accuracy between the three rules can be substantial.*

Accuracy of all rules and hence also differences are smaller for low levels of signal quality $q$ and they would vanish when quality approaches $q = 0.5$. Similarly, differences also vanish when link reliability approaches $r = 0.5$.

Let us now explore how the presence of multiple source nodes affects the comparison of the three rules. The simulations indicate that the shortest path rule may be closer to the random neighbor rule for one source node, but closer to the Bayesian rule when there are multiple source nodes. Comparing the left and right panel of the figure reveals that multiple source nodes improve accuracy of the Bayesian and the shortest path rule. For the random neighbor rule, multiple source nodes do not substantially improve accuracy. Recall that this rule effectively uses only one source node. Thus, multiple source nodes do not help to aggregate information, but only to reduce the length of the used path, on average. This disadvantage of the random neighbor rule can also be seen for the case of perfectly reliable links, $r = 1$, and understood as follows.

In the extreme case of $r = 1$, the signed network has the two-camp structure such that there are only positive links within camps and negative links across (Cartwright and Harary, 1956). In other words, the network is socially balanced (Harary et al., 1953). Hence, there are only two states of the world, which differ by which camp is of the positive type. Source nodes provide information about the likelihood of these two states. In particular, if there is only one source node ($N_S = 1$), the expected accuracy of any of the three rules is $q$. For more than one source node, the expected accuracy is higher for the Bayesian and the shortest path rule, but not for the random neighbor rule. Accuracy of the random neighbor rule stays at $q$, no matter how the number of source node changes. This is the intuition for the second result:

**Result 2.** *The difference in accuracy between the shortest path rule and the random neighbor rule is particularly pronounced for multiple source nodes.*

Moreover, we can observe Figure 5 that $r$ is not an upper bound for expected accuracy of the Bayesian rule. For instance at $r = 0.9$, the Bayesian rule's expected accuracy is larger than 0.9. The reason is that multiple paths with link reliability of $r$ can yield information that exceeds $r$. For the other two rules, the shortest path rule and the random neighbor rule, link reliability $r$ is in fact an upper bound for expected accuracy

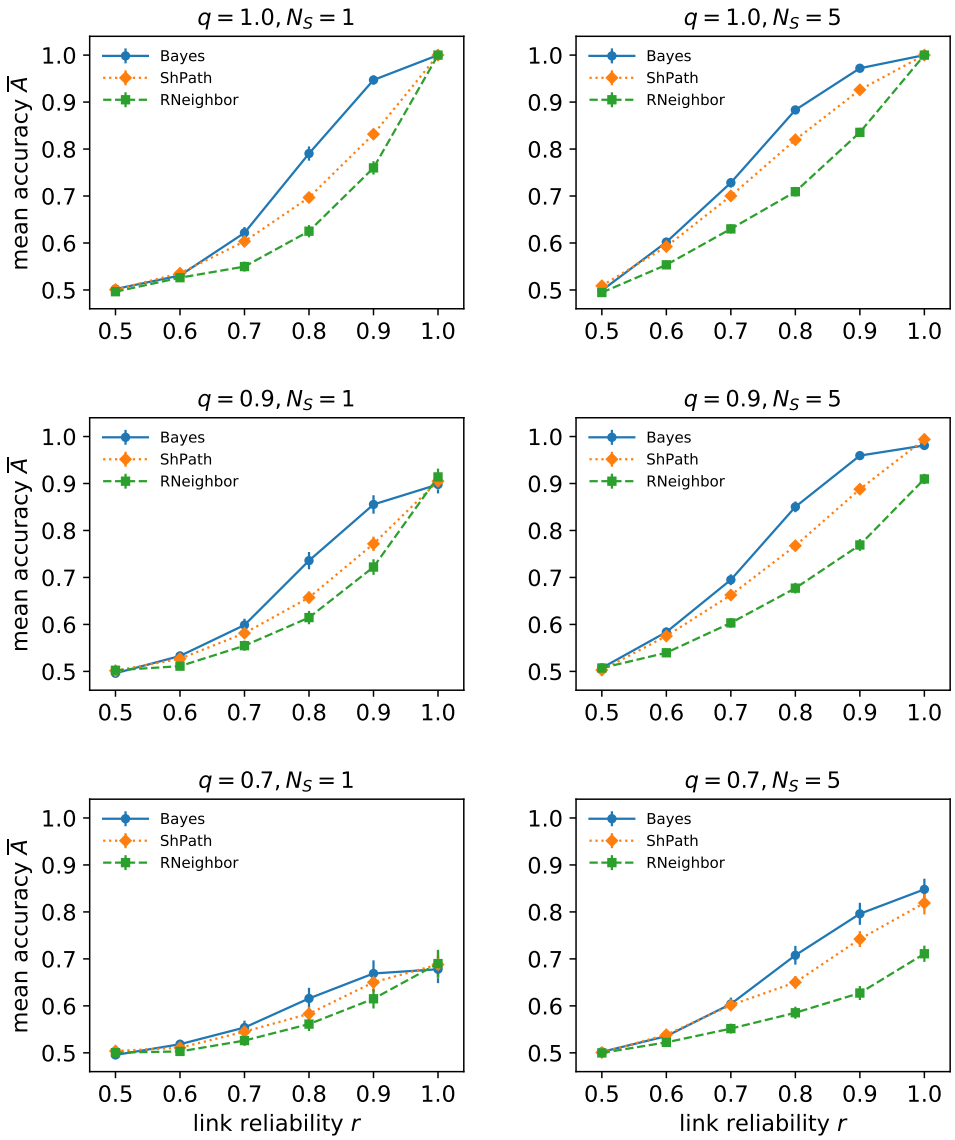


Figure 5: Mean accuracy of the three rules for different levels of link reliability $r$. Signal quality is $q = 1$ (top row), $q = 0.9$ (middle row), $q = 0.7$ (bottom row). Results are for Erdös-Rényi networks with $N = 20$, $z = 5$, $N_S = 1$ (left) and $N_S = 5$ (right). The error bars show twice the standard error of the mean.

when we have only one source node. The reason is that only one path is actually used such that information of multiple paths from one source node are not aggregated by these two rules (cf. Example 1).

## 4.3 Different Network Topologies

We now investigate how the network topology affects accuracy. First, we check whether our simulation results are robust to changing the class of network. We have so far used Erdös-Rényi (ER) random networks, now we consider three additional network classes: regular graphs (RG), Barabási-Albert (BA) scale-free networks, and Watts-Strogatz small world (WS) networks. These networks differ in several characteristics: shape of the degree distribution (regular graphs have the most homogeneous degree distribution, BA networks the most heterogeneous), clustering (WS networks have high clustering coefficient when the rewiring probability is small), and the distances, as we will discuss.

Figure C.1 in Appendix C.2 re-examines the differences between the three rules of Figure 5 when the Erdös-Renyi network generation process is replaced with RG, BA, and WS. Indeed, the differences between the three rules are very similar for all topologies and hence our two empirical results R1 and R2 (see Section 4.2) are robust.

Second, we address how two crucial features of a network structure – density and distances – affect accuracy.

Concerning density, Figure 6 shows how expected accuracy in the three rules increases with mean degree $z$. Intuitively, denser networks provide more information and shorter paths, which should increase expected accuracy of all three rules. We see that this is the case for the Bayesian rule and, to a lesser extent, for the shortest path rule. For the random neighbor rule, however, the increase of expected accuracy is not significant for $N_S = 5$ and it saturates early (at $z \approx 4$) for $N_S = 1$. This agrees with the results presented in Medo et al. (2020) where a weak dependence on $z$ was observed too. Hence, we conclude:

**Result 3.** *Accuracy of the Bayesian and shortest path rule weakly improve with density of a network.*

Let us now address the effect of distances. To vary the distances, without changing the density of a network, we use two ways. First, we vary the rewiring probability in the WS networks, in which distances are decreasing in the rewiring probability. Second, we study the different classes of networks – ER, RG, BA, WS – all for the same density and measure the distances their realizations have. It turns out that our simulated networks

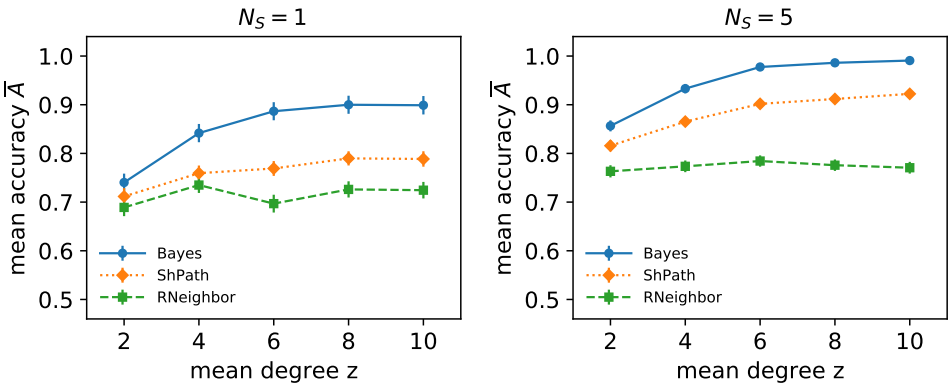


Figure 6: Mean accuracy of the three rules for increasing mean degree $z$. Results are for Erdös-Rényi networks with $N = 20$, $z = 5$, $q = 0.9$, $r = 0.9$, $N_S = 1$ (left) and $N_S = 5$ (right).

satisfy $d_{BA} < d_{ER} < d_{WS(\beta=0.5)} \approx d_{RG} < d_{WS(\beta=0.1)}$ (empirically determined shortest lengths are $3.54 \pm 0.03$, $4.20 \pm 0.03$, $4.51 \pm 0.02$, $4.55 \pm 0.01$, and $6.85 \pm 0.05$, mean $\pm$ twice the SEM, respectively) for $N = 300$ and $z = 4$ in Figure 7.

Figure 7 shows that the mean accuracy achieved by the shortest path rule is the lowest for the WS networks with low rewiring probability, $\beta = 0.1$, where the average shortest path length is considerably longer than in the other networks. By contrast, the highest mean accuracy (for fixed link reliability, $r$) is achieved in BA networks where the average shortest path length is the shortest. These observations agree with Claim 3 which identifies the shortest path length as the key determinant of the expected accuracy for the shortest path rule. These findings strongly suggest that the average path length of a network is a key determinant for accuracy of the shortest path rule.

**Result 4.** *The shorter the average distances in a network, the higher the accuracy of the shortest path rule.*

The same ordering of network topologies has been reported for the random neighbor rule in Medo et al. (2020). Figure C.2 in Appendix C.2 shows similar results for the Bayesian rule: Watts-Strogatz networks with $\beta = 0.1$ display the longest shortest paths and the lowest mean accuracy. This suggests that the shortest path length is important also for this optimal rule.

## 4.4 Increasing the Network Size

We finally analyze how results scale when network size $N$ becomes larger. To focus on the effect of growing network size only, we keep mean degree fixed. We consider two settings.

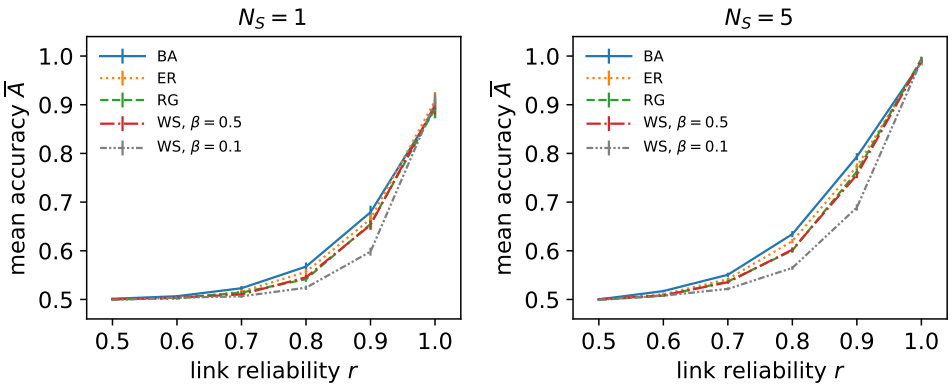


Figure 7: Accuracy of the shortest path rule for various classes of random networks and different levels of link reliability $r$. Results are for $N = 300$, $z = 4$, $q = 0.9$, $N_S = 1$ (left) and $N_S = 5$ (right).

Either the number of source nodes is kept constant or it grows proportionally to network size $N$. The dependence of the accuracy on the network size has been a major topic in Medo et al. (2020). They show that the random neighbor rule's expected accuracy slowly converges to 0.5 (equivalently, in their terms the rule's consistency convergences to 0) as $N$ grows when the number of source nodes is constant. When $N_S \sim N$, the expected accuracy of the random neighbor rule quickly approaches a limit value greater than 0.5. We now explore accuracy of the shortest path rule in comparison to the random neighbor rule for growing $N$ with simulations. We also include results for the Bayesian rule, although this exercise is limited, as the rule's computational complexity restricts our simulations to $N \leq 20$. We focus our discussion on settings with $r, q > 0.5$, where there is at least some opportunity to learn.

Figure 8 shows the mean accuracy against the network size for several settings of $q$ and $r$, using Erdös-Rényi random networks with mean degree $z = 5$. Comparing the different panels, mean accuracy in the top row is slightly lower than in the middle row because it has a lower signal quality of $q = 0.9$ (versus $q = 1$), while link reliability is $r = 0.9$ in both. Accuracy in the bottom row is lower than in the middle row because it has a lower link reliability $r = 0.7$ (versus $r = 0.9$), while signal quality is perfect, i.e. $q = 1$, in both.

Besides confirming the previously reported results for the random neighbor rule (Medo et al., 2020), there are a number of observations to make from Figure 8. First, the shortest path rule performs significantly better than the random neighbor rule, in particular when there are several source nodes ($N_S = 5$, middle column) or when the number of source nodes grows with the number of nodes ($N_S = N/10$, right column). This confirms Results 1 and 2 from above.

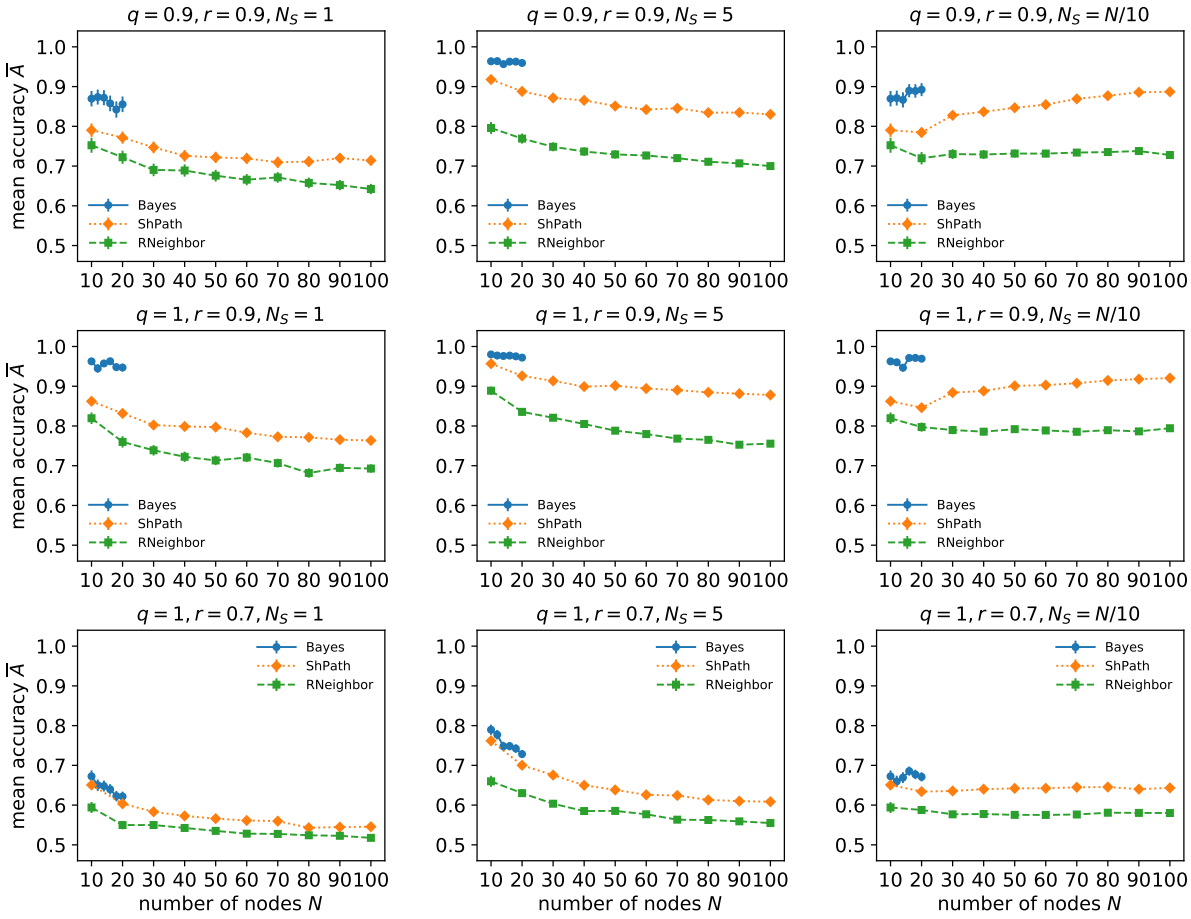


Figure 8: Mean accuracy of the three rules for growing network size $N$. Results are for Erdös-Rényi networks with $z = 5$, $q = 0.9$ and $r = 0.9$ (top row), $q = 1$ and $r = 0.9$ (middle row), and $q = 1$ and $r = 0.7$ (bottom row), $N_S = 1$ (left column), $N_S = 5$ (middle column), and $N_S = f_S N$ with $f_S = 0.1$ (right column).

Second, when $N_S$ is fixed, the mean accuracy of all three rules decreases with $N$, albeit at different rates. The rate of the shortest path rule, like for the random neighbor rule, can be derived analytically for a single source node, as detailed in Appendix C.3. In particular, the classical result for the average distance between two nodes chosen at random in Erdös-Rényi random networks, $\overline{L} \approx \ln N / \ln z$ (e.g. Bollobás, 2001), can be plugged in Eq. (7) to obtain

$$E[A^{ShPath,ER}] = \tfrac{1}{2} + \left(q - \tfrac{1}{2}\right)(2r-1)^{\ln N / \ln z} \tag{8}$$

which holds for the shortest path rule and one source node. This $E[A^{ShPath,ER}]$ approaches 0.5 as $N \to \infty$ for any $r < 1$. The reason for the decrease in expected accuracy is the effect of longer distances (cf. Result 4), but the speed of convergence to zero is lower than for the random neighbor rule.

**Result 5.** *When the number of source nodes is fixed, mean accuracy of the shortest path rule converges to 0.5 slower than mean accuracy of the random neighbor rule.*

Third, when the number of source nodes grows with $N$, the mean accuracy can even increase with $N$ (see results for the shortest path rule for $r = 0.9$ and $N_S = N/10$)—the information gained from a growing number of source nodes is sufficient to overweight the shortest paths that become longer, on average, as $N$ grows. This occurs for sufficiently high link reliability $r = 0.9$. Theoretically, this effect is clear for $r = 1$, where additional source nodes mean additional information without any loss due to long distances.

**Result 6.** *Given that link reliability $r$ is large enough, when the number of source nodes grows proportionally with network size $N$, accuracy of the Bayesian rule and the shortest path rule increases, in contrast to the random neighbor rule.*

Fourth, although the Bayesian rule produces substantially higher mean accuracy when $r = 0.9$, the difference between the Bayesian rule and the shortest path rule becomes much smaller when $r = 0.7$ (bottom row). This holds for all three considered numbers of source nodes: $N_S = 1$, $N_S = 5$, and $N_S = N/10$.

**Result 7.** *For low link reliability $r$ accuracy of all three rules remains relatively low regardless of the number of source nodes.*

# 5 Concluding Remarks

We have investigated how an observer can learn from a signed network where node types correlate with link signs. In such a setting, information about the type (e.g. trustworthiness) of source nodes is informative for the type of a target node. In stylized networks,

including trees, two relatively simple heuristics can lead to optimal learning, as we show in the theoretical part. Studying less-stylized random networks with a sequence of simulations, we find that differences in accuracy between the two heuristics and the optimal rule can be substantial. The heuristic that was introduced in the literature (Medo et al., 2020), the random neighbor rule, performs poorly when there is information about the type of multiple nodes. The heuristic that we introduce in this paper, the shortest path rule, performs much better with multiple source nodes, but generally relies on short average distances.

Our model could be extended in several directions. First, we focus on two node types, while we could integrate $k$ node types as follows. Instead of a single link reliability parameter $r$, consider a $k \times k$ matrix specifying the probability that nodes of each type are connected by a positive link. Our current model is then nested by using a two-by-two matrix where $r$ is on the diagonal and $1 - r$ is off the diagonal. Nevertheless, one justification for two node types comes from the property of structural balance that is equivalent to the two-camp structure (Harary et al., 1953).

Second, to keep our model simple, we assumed that positive and negative links are equally informative, as well as that positive and negative signals are equally informative. A more general model could introduce asymmetry, e.g. to capture that a positive signal form a source node with a positive link to the target node is more informative than a negative signal from a source node with a negative link.

Third, our model works with undirected networks. In some applications, the signed networks are directed and hence this is a considerable extension.

Fourth, one implicit assumption of our model is that the observer knows the whole network and all link signs. In particular, the application of the shortest path rule assumes that the observer has sufficient information to identify the shortest paths between nodes in the network which is a non-trivial assumption when the corresponding network is large. The research of navigability in complex networks (Boguna et al., 2009; Malkov and Yashunin, 2018) is relevant in this respect, as it seeks to understand which features of complex networks are helpful for finding shortest paths in them.

Finally, our model supposes that only link signs depend on node types, while the presence or absence of a link does not. This makes the analysis and interpretation clean, but it is a limitation, as in reality also the presence of a link can depend on link types. For instance, homophily in link formation can influence the presence of links (e.g. Currarini et al., 2009) and evidence on homophily is abundant (e.g. Jackson, 2019). Given homophily, an observer can learn from the sheer presence of links about the type of nodes of interest. With this perspective, the vast literature on detecting communities in non-signed networks (Fortunato and Hric, 2016) serves the goal of learning about node types from

the presence of links. Complementary to that we propose a model of learning about node types from the sign of links. A new research avenue would combine these two approaches in a framework that accommodates link presence and link signs that depend on link type.

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# 6 Acknowledgement

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# A Appendix: Proofs

## A.1 Proof of Claim 1 (Equivalence)

The condition of Claim 1 (a unique path between $t$ and each $s \in S$ such that these paths are non-overlapping) implies a network structure as exemplified in Figure 2 where $t$ is connected with source nodes by non-overlapping rays. Besides this tree structure, the network can contain arbitrary subgraphs appended to individual network nodes as they do not affect the uniqueness of paths between source nodes and $t$ (see the subgraph attached to node $j_3$ and the path between $s_5$ and $t$ in Figure 2, for example).

We first show that if the network consists only of the path between one source node, $s$, and target node, $t$, the Bayesian rule represented with Eq. (1) simplifies to

$$P[\boldsymbol{\theta}|\boldsymbol{\sigma}, \boldsymbol{R}] = q^{z_1(\boldsymbol{\theta})}(1-q)^{z_2(\boldsymbol{\theta})} r^{z_3(\boldsymbol{\theta})}(1-r)^{z_4(\boldsymbol{\theta})}$$

as the denominator of Eq. (1) (after canceling $1/2^N$) is then

$$\Omega := \sum_{\boldsymbol{\theta}' \in \Theta} q^{z_1(\boldsymbol{\theta}')}(1-q)^{z_2(\boldsymbol{\theta}')} r^{z_3(\boldsymbol{\theta}')}(1-r)^{z_4(\boldsymbol{\theta}')} = 1.$$

We prove this by induction. For the elementary graph with two nodes, $s$ and $t$, and one link, the sum over four possible states yields $\Omega = qr + q(1-r) + (1-q)r + (1-q)(1-r) = 1$. Here the first term, for example, corresponds to correct source signal and type-conforming link between the two nodes. In one induction step, we append a node of degree one to one

end of the current network for which $\Omega = 1$ holds (adding a link elsewhere would create a network outside the considered class). In the new denominator $\Omega'$ each term of the original denominator $\Omega$ is paired with two possible states of the added node, corresponding to forming a node-types conforming link (with probability $r$) and a nonconforming link (probability $1 - \delta$), respectively. We can thus write $\Omega' = \Omega r + \Omega(1 - r) = \Omega = 1$ which completes this step of the proof.

As the second step, we show that the belief derived with the Bayesian rule in Eq. (2) coincides with the belief derived with the shortest path rule in Eq. (5) when the network consists only of the shortest path between $s$ and $t$. Eq. (2) states that

$$b^{Bayes}(t) = \sum_{\boldsymbol{\theta} \in \Theta(t)} P[\boldsymbol{\theta}|\boldsymbol{\sigma}, \boldsymbol{R}]$$

where $\Theta(t)$ are all $2^{N-1}$ states with $\theta_t = +1$. When $\boldsymbol{\sigma}$ and $\boldsymbol{R}$ are given, the mapping between states of the world $\boldsymbol{\theta}$ and the values of $z_1$ and $z_2$ together with the conformity information for each link is bijective. The direction from $\boldsymbol{\theta}$ to $z_1$ and conforming/nonconforming links is obvious as it is used by the Bayesian rule. For the opposite direction, $\sigma_s$ combined with $z_1$ determines the type of $s$ (if $z_1 = 1$, $\theta_s = \sigma_s$, otherwise $\theta_s = -\sigma_s$). Progressing from the source node, the next node's state is $\theta_i = R_{ij}\theta_j$ if the link is conforming to the node types and $\theta_i = -R_{ij}\theta_j$ if the link is nonconforming. Each term in the sum in Eq. (2) thus appears also in Eq. (5) (and vice versa). When $o_{s \to t} = +1$, for example, terms proportional to $q$ (correct source signal) will be proportional to $(1 - r)^n$ where $n$ is even because an odd number of incorrect links would lead to $\theta_t = -1$ which does not contribute to the sum in Eq. (2). This step of the proof is thus concluded.

As the third step, we prove that attaching a subgraph to a single node in a given network does not change the result obtained with the Bayesian rule. Upon attaching a subgraph with link signs $\boldsymbol{R}'$ to a node, the space of all states grows from $\Theta$ to $\Theta \times \Psi$ where $\Psi$ is the space of states $\boldsymbol{\psi}$ of the subgraph nodes. We can write $P(\boldsymbol{\theta}|\boldsymbol{\sigma}, \boldsymbol{R}) = \sum_{\boldsymbol{\psi} \in \Psi} P(\boldsymbol{\theta}, \boldsymbol{\psi}|\boldsymbol{\sigma}, \boldsymbol{R}, \boldsymbol{R}')$. The numerator of $P(\boldsymbol{\theta}|\boldsymbol{\sigma}, \boldsymbol{R})$ can be written as

$$\frac{1}{2^{N+N'}} \cdot q^{z_1(\boldsymbol{\theta})}(1-q)^{z_2(\boldsymbol{\theta})} r^{z_3(\boldsymbol{\theta})}(1-r)^{z_4(\boldsymbol{\theta})} \sum_{\boldsymbol{\psi} \in \Psi} r^{z_3(\boldsymbol{\psi})}(1-r)^{z_4(\boldsymbol{\psi})}$$

where $N'$ is the number of nodes in the subgraph. Crucially, the sum over $\boldsymbol{\psi} \in \Psi$ is independent of $\boldsymbol{\theta}$ as the subgraph is attached to a single node. If the attachment node is of a positive type and $\boldsymbol{\psi}$ yields some $z_3(\boldsymbol{\psi})$ and $z_4(\boldsymbol{\psi})$, then $-\boldsymbol{\psi}$ would yield the same $z_3$ and $z_4$ for a negative type of the attachment node.[20] The terms $1/2^{N+N'}$ and

[20] If, by contrast, we attach node $x$ to two different nodes of an existing graph, the sum over two possible

$\sum_{\psi\in\Psi} r^{z_3(\psi)}(1-r)^{z_4(\psi)}$ cancel out and we find that the obtained $P(\boldsymbol{\theta}|S,\boldsymbol{R})$ is the same as without the attached subgraph. If adding subgraphs does not change the Bayesian inference, removing them has no effect as well. If only the shortest path between $s$ and $t$ remains after removing all subgraphs attached to single nodes, the Bayes rule thus reduces to the shortest path rule as we wanted to prove.

We have shown that for every network with a unique path from source node to target node both Eq. (2) and Eq. (5) correctly determine the posterior belief about the target node's type. Consider now a network with multiple source nodes that have unique, non-overlapping, paths to the target node as illustrated in Figure 2. Each subgraph with a source node yields independent information about the target node. Since Eq. (6) is the correct aggregation formula for independent signals, we can conclude that the introduced shortest path rule indeed coincides with the Bayes rule for the given class of networks, which finally completes our proof.

## A.2 Proof of Proposition 2 (Same Best Guess)

We first show that according to the shortest path rule, the best guess coincides with the path orientation, when there is only one source node. The shortest path rule uses path orientation defined by Eq. (3), $o_{s\to t} := \sigma_s \prod_{m=0}^{L-1} R_{n_m,n_{m+1}}$ where $n_0,\ldots,n_L$ are the nodes along the path, $n_0 := s$, and $n_L := t$. It further computes the probability that the path orientation agrees with the type of node $t$ as $P(o_{s\to t}=\theta_t) = q\pi_{s\to t} + (1-q)(1-\pi_{s\to t})$ (see Eq. (5) and the discussion thereafter). Here $\pi_{s\to t}$ is the probability that the path between $s$ and $t$ has an even number of errors. Since $r > 0.5$, it follows from Eq. (4) that $\pi_{s\to t} > 0.5$ for any finite $L$. We can now rewrite

$$P(o_{s\to t}=\theta_t) = 1 - \pi_{s\to t} + q(2\pi_{s\to t} - 1)$$

which grows with $q$. Since $P(o_{s\to t}=\theta_t) = 0.5$ when $q = 0.5$, $P(o_{s\to t}=\theta_t) > 0.5$ for $q > 0.5$. The computed probability that $o_{s\to t}$ correctly identifies the type of node $t$ is more than 0.5, so the best guess of $t$'s type produced by the shortest path rule is thus always the path orientation. Note that we assumed here one source node but not the path uniqueness, so the last statement holds for the shortest path rule on any network where $N_S = 1$.

When the path between $s$ and $t$ is unique, the random neighbor rule is deterministic as the path over which the opinion on $t$ is formed is given. According to this rule, the

states of node $x$ is $r^2 + (1-r)^2$ when the attachment nodes have the same types whereas it is $2r(1-r)$ when the attachment nodes have different types. The sum over $\boldsymbol{\psi} \in \Psi$ thus depends on $\boldsymbol{\theta}$ and does not factor out from $P(\boldsymbol{\theta}|\boldsymbol{\sigma},\boldsymbol{R})$.

judgment of node $n_1$, which is adjacent to $n_0 := s$, is a product of the source signal and the sign of the link between the two nodes. The judgment of node $n_2$ is further multiplied by $R_{n_1 n_2}$, and so on, until the judgment of $t$ is determined as a product of the source signal with link signs along the path which is precisely the path orientation $o_{s\to t}$. The two rules thus coincide when $N_S = 1$ and the path between $s$ and $t$ is unique.

## A.3 Proof of Claim 3 (Shortest Path Accuracy)

Assume that node states $\boldsymbol{\theta}$ are fixed. As we showed in Section A.2, $o_{s\to t}$ is the best guess of $\theta_t$ for the shortest path rule. The probability that $o_{s\to t} = \theta_t$ is $q\pi_{s\to t} + (1-q)(1-\pi_{s\to t}$ where the first term accounts for a correct source signal *and* an even number of errors along the path from $s$ to $t$, and the second term accounts for an incorrect source signal *and* an odd number of errors along the path from $s$ to $t$, respectively. Using Eq. 4, this probability can be simplified to the form

$$P(o_{s\to t} = \theta_t) = \tfrac{1}{2} + \left(q - \tfrac{1}{2}\right)(2r-1)^L.$$

This result already takes into account the limited reliability of source signals and link signs: it averages accuracy over different realizations of $\boldsymbol{\sigma}$ and $\boldsymbol{R}$. As it does not depend explicitly on node states $\boldsymbol{\theta}$, it is straightforward to average it over $\boldsymbol{\theta}$ and thus obtain the expected accuracy in line with Definition 2.

## A.4 Proof of Proposition 1 (Ordering)

The Bayesian rule uses all available information, source signals $\boldsymbol{\sigma}$ and link signs $\boldsymbol{R}$, to correctly compute the objective probability that target node $t$ is of positive type. It is thus the optimal rule by construction and its expected accuracy is the highest of all possible rules, hence $E[A^{Bayes}] \geq E[A^{ShPath}]$ and $E[A^{Bayes}] \geq E[A^{RNeighbor}]$.

When $N_S = 1$, the ordering between the shortest path rule and the random neighbor rule can be established as follows. Denote the shortest path length between $s$ and $t$ as $L$; the expected accuracy of the shortest path rule can be obtained using Eq. (7). As already explained, the judgment formed using the random neighbor rule corresponds to a path between $s$ and $t$. Denote this path's length as $L'$. As the best guess of the shortest path rule is the same as the judgment formed formed by the random neighbor rule (both are equal to the path orientation $o_{s\to t}$), the expected accuracy of the random neighbor rule can be computed using Claim 3 with the only difference being that the shortest path length $L$ is replaced by the actual path length $L'$. Now, every path length $L'$ must be

at least as long as the shortest path length, $L' \geq L$. As the expected accuracy decreases with $L'$ [when $r \in (0.5, 1)$], we immediately obtain that $E[A^{ShPath}] \geq E[A^{RNeighbor}]$.

When $N_S > 1$, the judgment of $t$ formed using the random neighbor rule is still formed using a path of length $L'$ between one source node and target node $t$. By contrast, the shortest path rule uses shortest paths from each source node and aggregates beliefs resulting from Eq. (5) using Eq. (6). While these paths have different lengths, $L_i$, the shortest of them cannot be longer than $L'$: $\min L_i \leq L'$. This path alone results in expected accuracy of the shortest path rule which is larger or equal than expected accuracy of the random neighbor rule. The additional paths used by the shortest path rule, some of which may be longer than $L'$, represent useful information that is taken into account by this rule and thus further improve its expected accuracy. We can thus conclude that $E[A^{ShPath}] \geq E[A^{RNeighbor}]$ holds also for $N_S > 1$.

### A.5 Proof of Proposition 2 (Unique Path)

According to Claim 1, a unique non-overlapping path between every source node $s$ and the target node $t$ implies that the best guess obtained with the Bayesian rule and the shortest path rule are the same. If, in addition, there is only one source node, Claim 2 states that the best guess of the shortest path rule and the random neighbor rule are the same. When $N_S = 1$ and the path between $s$ and $t$ is unique, the three rules thus yield the same best guess for any realization of $\boldsymbol{\theta}$, $\boldsymbol{\sigma}$, $\boldsymbol{R}$, their accuracy values are thus the same. As a consequence, the expected accuracy values obtained by averaging over $\boldsymbol{\theta}$, $\boldsymbol{\sigma}$, $\boldsymbol{R}$ are also the same.

## B Appendix: Examples

### B.1 Application of the Three Rules to the Example Depicted in Figure 1

Reconsider the very first example of employees that is depicted in Figure 1. In terms of our set-up vertices are $V = \{s_1, s_2, i, j, t\}$, where $s_1$ and $s_2$ are the source nodes and $t$ stands for the target node. Edges are $E = \{s_1t, s_1i, s_2j, it, jt\}$ with signs $R_{s_1t} = R_{it} = R_{s_2j} = +1$ and $R_{s_1i} = R_{jt} = -1$. Finally, source nodes' signals are $\sigma_{s_1} = +1$ and $\sigma_{s_2} = -1$ and (by convention $s_i = s_j = s_t = 0$). We have addressed two subgraphs of this graph as Examples 1 and 2 in the main text.

Now, for the complete example, the Bayesian rule considers $2^5 = 32$ states. The result

simplifies to

$$b^{Bayes}(t) = \frac{2r^2(1-r) + q(1-4r+3r^2+2r^3-r^4) + q^2(-1+4r-3r^2-4r^3+5r^4)}{2r^2(2-3r+2r^2) + q(2-8r+7r^2+6r^3-9r^4) + q^2(-2+10r-15r^2+4r^3+5r^4)}$$

and $b^{Bayes}(t) \approx 0.748$ when $q = 0.9$ and $r = 0.8$.

The shortest path rule combines the information from both parts of the network by applying Equation (6) to the beliefs that were derived from the paths $s_1, t$ and $s_2, j, t$. This yields

$$b^{ShPath}(t) = \frac{b_{s_1 \to t} b_{s_2 \to t}}{b_{s_1 \to t} b_{s_2 \to t} + (1 - b_{s_1 \to t})(1 - b_{s_2 \to t})}.$$

When $q = 0.9$, $r = 0.8$, then $b_{s_1 \to t} = 0.74$, $b_{s_2 \to t} = 0.644$, and $b^{ShPath}(t) \approx 0.837$. Observe that the belief from the path $s_1, t$ is stronger than from the longer path $s_2, y, t$, which illustrates that the shortest path rule discounts paths by their length. Beside this, aggregating information from both paths improves the final confidence level. This reinforcement of beliefs holds here as both paths have the same orientation.

The random neighbor rule starts with judging the two source nodes. The judgement of $t$ depends on the realization of the random order of nodes to be judged. Overall, we have a 29/36 chance to reach $g^{RNeighbor}(t) = +1$ and a 7/36 chance to reach $g^{RNeighbor}(t) = -1$.

## B.2 More Examples

**Example B.1** (Two Triangles)**.** *Let* $(V, E) = (\{s, j_1, j_2, t\}, \{st, sj_1, sj_2, j_1t, j_2t\})$. *Let* $R_{st} = R_{sj_1} = R_{sj_2} = +1$ *and* $R_{j_1t} = R_{j_2t} = -1$ *and let* $\sigma_s = +1$, *as illustrated in Panel (A) of Figure 3. The Bayesian rule yields:*

$$b^{Bayes}(t) = \frac{4qr^3(1-r) + 2(1-q)r^2(1-r)^2 + (1-q)[r^4 + (1-r)^4]}{4r^3(1-r) + 2r^2(1-r)^2 + [r^4 + (1-r)^4]}$$

*where* $b^{Bayes}(t) \approx 0.476$, *when* $q = 0.9, r = 0.8$. *The shortest path rule yields*

$$b^{ShPath}(t) = qr + (1-q)(1-r)$$

*where* $b^{Bayes}(t) = 0.74$, *when* $q = 0.9, r = 0.8$.

*We receive* $b^{Bayes}(t) < b^{ShPath}(t)$ *(for any* $q, r \in (0.5, 1)$*), which can be checked, e.g. by Mathematica 12.0. The shortest path rule only uses the direct path, which has a positive orientation. However, the Bayesian rule uses all information including the two paths of length 2, which are both negatively oriented.*

**Example B.2** (Multiple shortest paths)**.** *Let* $(V, E) = (\{s, j_1, j_2, j_3, t\}, \{sj_1, sj_2, sj_3, j_1t, j_2t, j_3t\})$.

*Let $R_{ij} = +1$ for all links in $E$ except for $j_3t$ ($R_{j3t} = -1$) and $\sigma_s = +1$, as illustrated in Panel (B) of Figure 3.*

*The Bayesian rule always leads to a belief $b^{Bayes}(t) > 0.5$, hence the best guess is $g^{Bayes}(t) = +1$.*

*The shortest path rule selects one of the three shortest paths at random. Hence, with probability 2/3 it induces belief $b^{ShPath}(t) = qr + (1-q)(1-r) > 0.5$ and with probability 1/3 it induces belief $b^{ShPath}(t) = q(1-r) + (1-q)r < 0.5$.*

**Example B.3** (Overlapping Paths)**.** *Let $(V, E) = (\{s_1, s_2, j, t\}, \{s_1j, s_2j, jt\})$. Let $R_{ij} = +1$ for $ij \in E$ and let $\sigma_{s_1} = \sigma_{s_2} = +1$, as illustrated in Panel (C) of Figure 3.*

*The Bayesian rule yields:*

$$b^{Bayes}(t) = \frac{q^2[r^3 + (1-r)^3] + 2q(1-q)r(1-r) + (1-q)^2r(1-r)}{[q^2 + (1-q)^2][(1-r)^3 + r^3] + 4q(1-q)r(1-r) + [q^2 + (1-q)^2]r(1-r)},$$

*which is $b^{Bayes}(t) \approx 0.734$ for $q = 0.9$ and $r = 0.8$. The shortest path rule yields:*

$$b^{ShPath}(t) = \frac{b_{s_1 \to t} b_{s_2 \to t}}{b_{s_1 \to t} b_{s_2 \to t} + (1 - b_{s_1 \to t})(1 - b_{s_2 \to t})}$$

*where $b_{s_1 \to t} = b_{s_2 \to t} = q[r^2 + (1-r)^2] + 2(1-q)r(1-r)$, for $q = 0.9$ and $r = 0.8$, $b_{s_1 \to t} = b_{s_2 \to t} = 0.644$. Finally, we get $b^{ShPath}(t) \approx 0.766$. We receive (for any $q, r \in (0.5, 1)$): $b^{ShPath}(t) > b^{Bayes}(t)$, which can be checked, e.g. by Mathematica 12.0.*

# C Appendix: Simulations

## C.1 Simulation Details

*Erdös-Rényi* (ER) random networks have two parameters: network size $N$ and probability $p$ which determines the connection probability for every pair of nodes in the network. We set $p = z/(N-1)$ to obtain mean node degree $z$.

*Regular graphs* (RG) are random graphs where each node has $z$ links. They are constructed by assigning $z$ "stubs" to each node and matching node stubs at random until no unmatched stubs remain. Loops and multiple links are avoided.

*Barabási-Albert* (BA) random networks with $N$ nodes are grown by gradually adding nodes to an initial empty network with $m$ nodes. Every new node with $m$ edges is attached to existing nodes using the preferential attachment mechanism (i.e., the probability of choosing a node is directly proportional to the node's degree). We set $m = [(n - \sqrt{n^2 - 2zn})/2]$ to obtain mean node degree $z$ (approximately).

*Watts-Strogatz* (WS) random networks have three parameters: network size $N$, mean degree $z$ (this is assumed to be an even integer), and rewiring probability $\beta \in [0, 1]$. The networks are generated as follows:

1. Construct a regular ring lattice where each node is connected to its $z$ nearest neighbors.

2. Go over all nodes $i = 1, \ldots, N$ and with probability $\beta$ rewire each of the links to their right neighbors (i.e, neighbors $j = (i + k) \mod N$ where $k = 1, \ldots, z/2$) to a node chosen at random. Loops and duplicate links are avoided.

In our simulations, we use the network generator functions provided by the LightGraphs v1.3.3 package of Julia programming language.[21]

[21] `https://github.com/JuliaGraphs/LightGraphs.jl`.

## C.2 Robustness of Simulations with respect to Network Topology

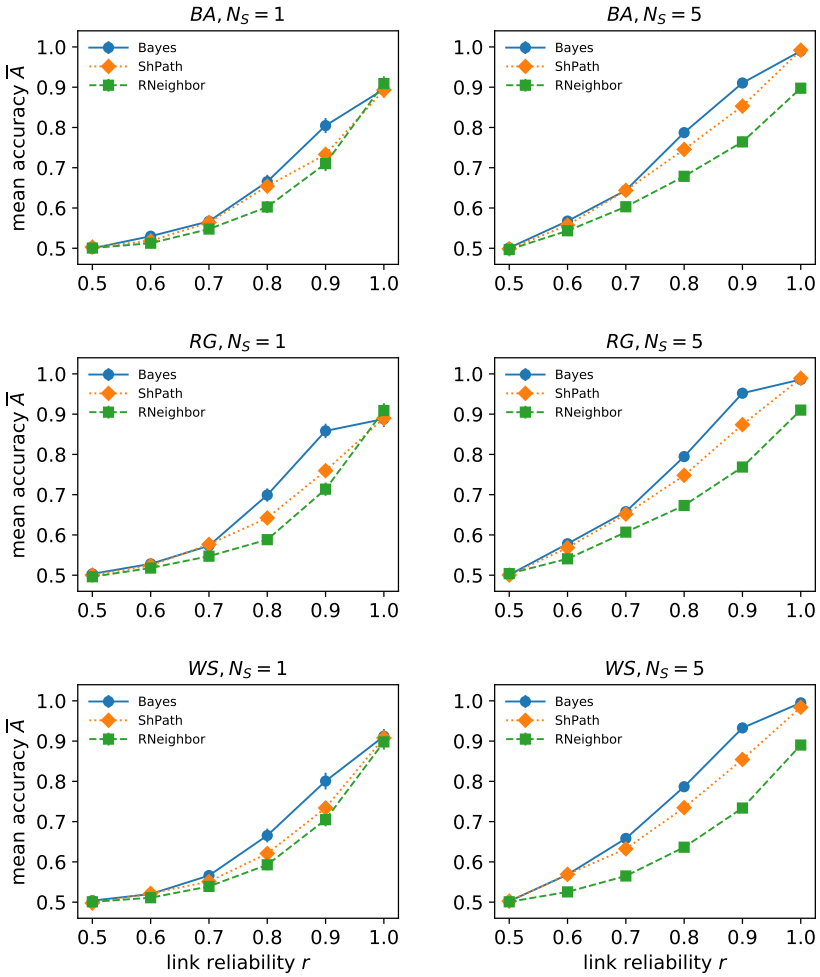


Figure C.1: Mean accuracy of the three rules for various classes of random networks. Setting signal quality $q = 0.9$, network size $N = 20$, average degree $z = 4$, and for different values of link reliability $r$. Top panels use BA, center panel RG, lower panel WS (rewiring probability is $\beta = 0.1$). Left panels have single source node; right panels have $N_S = 5$ source nodes.

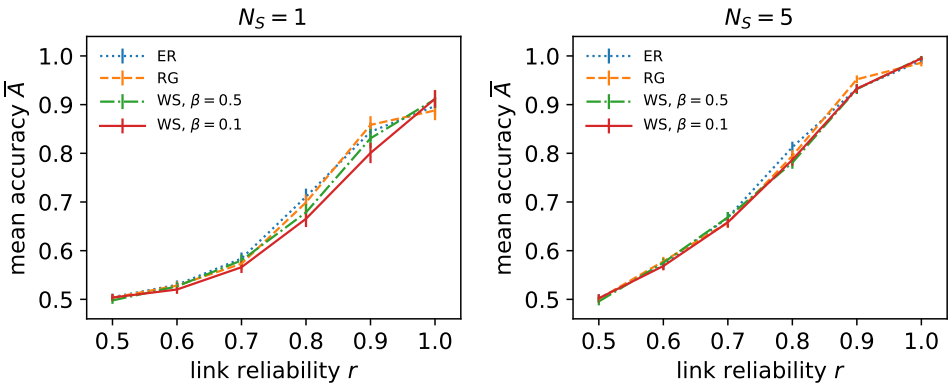


Figure C.2: Accuracy of the Bayesian rule for various classes of random networks and different levels of link reliability $r$. Results are for $N = 20$, $z = 4$, $q = 0.9$, $N_S = 1$ (left) and $N_S = 5$ (right). Shortest path lengths estimated in simulations for $N_S = 1$ are $2.17 \pm 0.02$ (ER), $2.16 \pm 0.01$ (RG), $2.18 \pm 0.01$ (WS, $\beta = 0.5$), and $2.51 \pm 0.02$ (mean $\pm$ twice the SEM) which qualitatively agree with the ordering of mean accuracy in the left panel. This suggests that the shortest path length is also important for accuracy of the Bayesian rule.

## C.3 Shortest Path Rule's Accuracy in Erdös-Rényi Random Networks

As explained in the main text, the average shortest distance between two random nodes in Erdös-Rényi random networks is $\overline{L}_{ER} \approx \ln N / \ln z$ (e.g. Bollobás, 2001). By plugging this $\overline{L}_{ER}$ for $L$ in Eq. (7), we obtain the expected accuracy of the shortest path rule in Erdös-Rényi random networks (Eq. (8) in the main text). As shown in Figure C.3, this analytical result agrees well with numerical simulations.

In (Medo et al., 2020), a relation similar to Eq. (8) has been derived for the random neighbor rule. This relation states that the "effective" accuracy, $E(A) - 0.5$, of the shortest path rule is proportional to $N^{-2(1-r)}$ when $N_S$ is fixed. Eq. (8) allows us to derive here a similar scaling relation for the shortest path rule. We find that $E[A^{ShPath}] - 0.5 \sim N^{-\gamma_{ShPath}}$ where

$$\gamma_{ShPath} = -\ln(2r - 1) / \ln z. \tag{C.1}$$

We see that unlike for the random neighbor rule where the scaling exponent $2(1 - r)$ depends only the link reliability (see Medo et al., 2020, for a derivation), here both link reliability and mean degree contribute. As mean degree increases, the scaling exponent $\gamma$ decreases which means that $E[A^{ShPath}] - 0.5$ vanishes slower with $N$, i.e. $E[A^{ShPath}]$ approaches 0.5 slower. At the same time, $\gamma_{ShPath}$ is independent of $q$: While signal quality $q$ strongly influences the absolute value of expected accuracy $E[A^{ShPath}]$, it does not affect

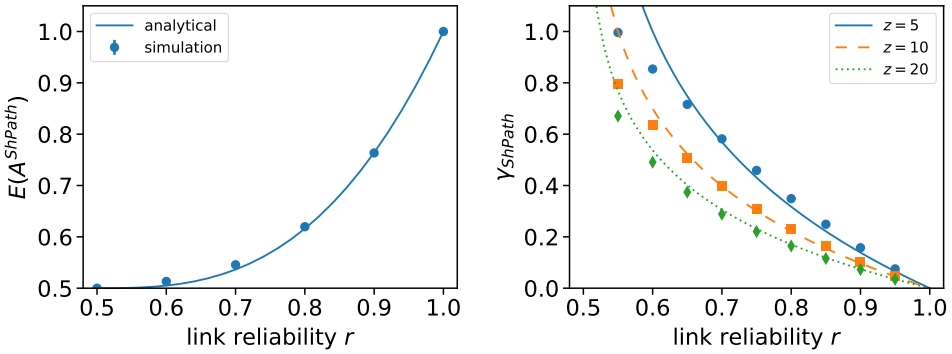


Figure C.3: (a) Left panel: Accuracy of the shortest path rule for Erdös-Rényi networks with $N = 100$, $z = 5$, $q = 1$. (b) Right panel: Estimated exponents of the scaling relation $E(A^{ShPath}) - 0.5 \sim N^{-\gamma_{ShPath}}$ for numerical simulations on Erdös-Rényi networks with various mean degree values (symbols) and analytical result obtained using Eq. (C.1) (lines). The exponents were estimated from simulation results using least squares fits between $\ln[E(A^{ShPath}) - 0.5]$ and $\ln N$ for $N$ between 50 and 500.

its decay with $N$.

Figure C.3b shows that while Eq. (C.1) agrees well with numerical simulations for $r \gtrsim 0.65$, significant deviations appear for low $r$ values (in particular, the analytical $\gamma$ diverges as $r \to 0.5$). The main reason for the deviations is that to derive Eq. (8), we plugged the average shortest path length in Eq. (7). While this is justified by relatively narrow distributions of the shortest path length in Erdös-Rényi networks, it does not produce sufficiently precise results when $r$ is close to 0.5: the term $2r - 1$ in Eq. (7) is then small and $A^{ShPath}$ thus depends strongly on $L$. If instead of using the average shortest path length, we numerically determine the distribution of the shortest path lengths in Erdös-Rényi networks and use this empirical distribution to average over various shortest path lengths in Eq. (7), we obtain scaling exponents that do not diverge as $r \to 0.5$ and agree well with numerical simulations for all $r$ (results can be requested from the authors).