Simulating the Law of Value

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Abstract

A computational model of a simple commodity economy is examined and a theory of the relationship between commodity values, market prices and the efficient division of social labour is developed. The main conclusions are: (i) the labour value of a commodity is an attractor for its market price; (ii) market prices are error signals that function to allocate the available social labour between sectors of production; and (iii) the tendency of prices to approach labour values is the monetary expression of the tendency of a simple commodity economy to efficiently allocate social labour. The results demonstrate the importance of Marx's law of value for understanding the dynamics of price and labour allocation in commodity economies.

1 Introduction

A fundamental building block of Marx's economics is the 'law of value'. The aim of this paper is to investigate this law by studying the dynamics of a computational model of a simple commodity economy.

2 The law of value

Marx, following Ricardo, held a labour theory of the economic value of reproducible commodities. The value of a commodity is determined by the prevailing technical conditions of production and measured by the socially necessary labour-time required to produce it (Marx (1954)). The value of a commodity is to be distinguished from its price, which is the amount of money it fetches in the marketplace.

According to Marx, although economic actors may differ in their subjective evaluations of the worth or 'value' of commodities, there are emergent regularities in commodity economies that ensure that prices tend to 'gravitate' around labour values. Subjective determinations of market prices are constrained by the prevailing commodity values, which are determined by the objective conditions of production.

In a theoretical simplification of capitalism often referred to as the 'simple commodity economy', Marx held that prices would tend to 'correspond' to labour values. Only a few simple conditions need be met for this to occur:

> For prices at which commodities are exchanged to approximately correspond to their values, nothing more is necessary than 1) for the exchange of the various commodities to cease being purely accidental or only occasional; 2) so far as direct exchange of commodities is concerned, for these commodities to be produced on both sides in approximately sufficient quantities to meet mutual requirements, something learned from mutual experience in trading and therefore a natural outgrowth of continued trading; and 3) so far as selling is concerned, for no natural or artificial[1] monopoly to enable either of the contracting sides to sell commodities above their value or to compel them to undersell. By accidental monopoly we mean a monopoly which a buyer or seller acquires through an accidental state of supply or demand.
>
> The assumption that the commodities of the various spheres of production are sold at their value merely implies, of course, that their value is the centre of gravity around which their prices fluctuate, and their continual rises and drops tend to equalise (Marx (1972, p. 178)).

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[1] Presumably this is a minor translation error that should be replaced with 'accidental'.

The theory of the law of value motivates such statements. It explains how the total labour of a society of commodity producers, who freely exchange their products in a marketplace, is divided and allocated to different branches of production via the market mechanism. The exchange of commodities at prices that deviate from values is the mechanism by which social labour-time is transferred from one sector of production to another. When prices equal values the division of labour has reached an equilibrium that satisfies social demand: 'the law of value is the law of equilibrium of the commodity economy' (Rubin (1973)):

> [I]t is only through the 'value' of commodities that the working activity of separate independent producers leads to the productive unity which is called a social economy, to the interconnections and mutual conditioning of the labour of individual members of society. Value is the transmission belt which transfers the movement of working processes from one part of society to another, making that society a functioning whole (Rubin (1973, p.81)).

In brief, the law of value is the process by which a simple commodity economy (i) reaches an equilibrium, in which (ii) prices correspond to labour values, and (iii) social labour is allocated to different branches of production according to social demand (where 'social demand' is understood to mean consumption requirements constrained by income).

The law is investigated by building a computational model of a simple commodity economy. It will be seen that the law of value does operate as described and that the model reveals particularly simple and satisfying dynamic relationships between values, prices, social labour-time and money.

It must be emphasised, however, that Marx did not state that prices correspond to labour values in capitalism. Instead, he thought there was a systematic casual relationship, mediated by the law of value, between labour values and profit-equalising 'prices of production'. This paper does not directly touch on these larger issues. The aim is to examine the law of value in a simple commodity economy, a model implicit in part one of the first volume of *Capital*, referred to by Marx as 'simple circulation', and elaborated by Rubin (1973).

A 'pure labour' model is developed and by definition the only constituent of production time is labour-time. No arguments are adduced in favour of assuming that labour-time is the substance of value in the presence of other mechanisms that produce use-value, such as machines or natural processes. For readers opposed to this assumption the term 'labour-time' can be replaced by the term 'production-time' throughout.

3 Methodology

This section contains some brief methodological remarks on complex systems, abstract generative mechanisms and stochastic modelling. The methodological issues concerning the use of computational models in the social sciences have been well documented (e.g. Epstein and Axtell (1996)) and are therefore not considered.

3.1 Complex systems

The simple commodity economy is simple compared to capitalism but nevertheless represents a complex system composed of many parts.

Approaches to modelling complex systems can be broadly categorised as deep-but-narrow (DBN) or broad-but-shallow (BBS). The DBN approach isolates a part of the system and models it in detail. For example, a DBN approach to modelling a commodity economy might isolate the market for analysis and model the kinds of bargaining rules employed by market participants, while abstracting from production, consumption and labour allocation. In contrast, the BBS approach considers the system as a whole but models constituent parts very simply. Modelling effort is concentrated on the causal relations between parts. The choice of approach depends on the modelling goals. Both are necessary to model a broad and deep economic reality.

The law of value is a phenomenon that emerges from multiple local interactions. There isn't a law of value mechanism in the simple commodity economy, the simple commodity economy is the mechanism of the law of value. A BBS approach is therefore adopted: the aim is to understand the emergent properties of a commodity economy considered as a whole (see Wright, Sloman and Beaudoin (1996) and Wright (1997) for this approach in the context of modelling the mechanisms underlying human emotions). All parts of a simple commodity economy – production, exchange using money, and consumption – are modelled, although in a simple manner. The heuristic principle employed is that the coarse-grained architecture of a complex system conditions its emergent dynamics: the *architecture dominates mechanism* (Sloman (1994)).

3.2 Abstract generative mechanisms

The simple commodity economy is a simplified model of capitalism and therefore it is necessary to provide some comments to justify its potential relevance.

Extending a broad-but-shallow model to more accurately reflect economic reality involves iterative deepening of its constituent mechanisms. The architecture may remain constant during this process if the initial model is sufficiently broad. For example, the simple commodity economy lacks many essential features of capitalism, such as economic actors who employ others and seek profit, input-output relationships between production sectors, and the use of machines in addition to human labour. Deeper models should include these features (and others) but can do so within the architectural framework of the simple commodity economy. New features generate new model properties yet the global dynamics remain conditioned by the original architecture. The task is to determine how the introduction of deeper mechanisms affects conclusions obtained from models with shallower mechanisms. The earlier conclusions inform the study of the deeper model, which otherwise may represent an intractable starting point, particularly if deeper mechanisms elaborate the dynamics such that lawful relationships arising from the abstract architecture are obscured.

It is for these reasons, although expressed in Hegelian terminology, that Marx viewed the simple commodity economy not only as an abstraction, but as potentially referring to a real and existing generative mechanism (see Sloman (1978) and Bhaskar (1997)) present in all commodity economies that inherit its architectural structure. The expectation is that more realistic models of capitalism share features with the original exemplar because it remains present within them. Of course, whether this is true or not ultimately depends on the explanatory and predictive consequences of the research project.

3.3 Stochastic models

Given that the constituent parts of the simple commodity economy are to be modelled very simply the question naturally arises how this is to be done.

Stochastic mechanisms that select outputs from a bounded set according to a probability distribution are employed to model processes that involve subjective indeterminacy (for example, local price evaluations) or elements of chance (for example, the matching of buyer and seller in the marketplace). The probability distributions are chosen to be uniform in accordance with Bernoulli's Principle of Insufficient Reason, which states that in the absence of knowledge to the contrary assume that all outcomes are equally likely. As a result, many variables of interest (for example prices and labour allocations) exhibit stationary distributions, rather than stationary values. Much of economic theory abstracts from distributions and theorises long-run average values, and Marx is no exception. The translation between stationary distributions and 'long-run' averages does not, in this case at least, present any problems.

4 The computational model

The model consists of a set of N economic actors (labelled $1, \ldots, N$) that produce, consume and exchange a set of L commodity types (labelled $1, \ldots, L$), a fixed amount of paper money M, which is distributed amongst the actors, a market mechanism that mediates commodity and money exchange, and a simulator that processes events and increments time.

4.1 Actor production

Every actor specialises in the production of a single commodity at any one time. The current specialization of actor i is given by $A(i)$. All commodities are simple, and do not require other commodities for their manufacture. Each commodity requires the work of a single actor for its production. Constant returns to scale prevail and consequently there is no rationale for the existence of firms. Actors never cease production. A production vector, $\mathbf{p} = (1/p_1, \ldots, 1/p_L)$, where $p_i > 0$, defines the rate at which an actor can produce each commodity type. For example, an actor that specialises in commodity type i produces at a rate of $1/p_i$ units per unit time. The production vector is identical and fixed for all actors. Labour in the economy is therefore homogenous and is not subject to changes in technique. Once a commodity is produced it remains the property of the actor until consumed or exchanged. Each actor has an associated endowment vector that represents how much of each commodity is currently held.

Actors produce according to the following rule:

Production update rule $\mathbf{P_1}$: (Deterministic). At the start of the simulation initialise the endowment vector for actor i to zero: $\mathbf{e}_i = \mathbf{0}$.

Actor i subsequently generates one unit of commodity $A(i)$ every $p_{A(i)}$ time steps, and the appropriate element of the endowment vector, $\mathbf{e}_i[A(i)]$, is incremented by one.

Although no producer is more efficient than another a distinction between socially necessary labour-time and actual labour-time expended can be maintained. Overproduction of a commodity relative to the social demand implies that some of the labour-time expended was socially unnecessary.

4.2 Actor consumption

Every actor desires to consume all commodity types. This behaviour can be interpreted as subsistence or aspirational. A consumption vector, $\mathbf{c} = (1/c_1, \ldots, 1/c_L)$, where $c_i \geq 0$, defines the desired rate of consumption events for all actors. For example, every actor desires to consume commodity i at a rate of $1/c_i$ units per unit time. The consumption vector is identical and fixed for all actors and represents an economy with homogenous tastes that do not change. Note the asymmetry between production rates and consumption rates: an actor always meets its single production rate, but only conditionally meets its consumption rates. Actual consumption rates depend on the availability of commodities produced by other actors.

Actors consume according to the following rule:

Consumption update rule $\mathbf{C_1}$: (Deterministic). At the start of the simulation initialise the consumption deficit vector for actor i to zero: $\mathbf{d}_i = \mathbf{0}$.

Actor i subsequently generates one unit of consumption deficit for each commodity $j = 1, \ldots, L$ every c_j time steps, and the appropriate element of the deficit vector, $\mathbf{d}_i[j]$, is incremented by one.

Each time step actor i consumes $\mathbf{o}_i = \min(\mathbf{e}_i, \mathbf{d}_i)$ commodities from its endowment to satisfy its current consumption deficit. A new endowment vector, $\mathbf{e}'_i = \mathbf{e}_i - \mathbf{o}_i$, and a new deficit vector, $\mathbf{d}'_i = \mathbf{d}_i - \mathbf{o}_i$ are formed.

The assumption of universal and constant production and consumption vectors could be relaxed by introducing supply and demand noise due to heterogeneity of consumption tastes and production efficiency, but this extension is not pursued here,

4.3 The reproduction coefficient

The reproduction coefficient, $\eta = \sum_{i=1}^{L} p_i/c_i$, measures whether, given the 'social facts' of the production and consumption vectors, the economy may realise an overall social surplus, deficit or balance. A value of $\eta = 1$ implies the economy can achieve a state of simple reproduction (where total production equals total consumption), $\eta > 1$ implies an economy permanently in overall deficit (unrealised consumption capacity) and $\eta < 1$ implies an economy permanently in overall surplus (redundant production capacity). The analysis is restricted to economies with $\eta = 1$ that can theoretically achieve a balance between supply and demand but may over-and under-produce commodities due to a sub-optimal division of labour.

4.4 Money

Each actor i owns a sum of symbolic money, $m_i \geq 0$, which is used to purchase commodities for consumption. The total amount of money in the economy, $M = \sum_{i=1}^{N} m_i$, is conserved. The unit of measure of money is the 'coin', although it is an arbitrarily divisible unit. Coins are neither produced nor consumed by actors. Actors exchange money for commodities, and therefore gain money when they sell, and lose money when they buy. Complications due to changes in the money supply are ignored.

4.5 Subjective prices

Actors form subjective evaluations of commodity prices during bi-lateral exchange. Two requirements are placed on the evaluations: (i) a purchaser cannot offer more coins than they possess, and (ii) offer prices must not be fixed *a priori*. The second requirement is important because the law of value trivially does not hold in an economy of homogenous, *a priori* evaluators. For example, if every actor evaluated commodity i at 0 coins for all time then prices cannot converge to labour values. The law of value operates 'behind the backs' of economic actors because they adapt to changing local circumstances that are not of their own choosing but the result of global system properties. A natural approach to satisfying the requirement for adaptivity is to employ an adaptive algorithm from the field of machine learning that has some psychological plausibility and functions to minimize the consumption error. But this is an unnecessary level of detail at present. Instead, actors form selling and buying prices for each commodity according to:

> Price offer rule $\mathbf{O_1}$: (Stochastic). The price of commodity j according to actor i is $k_j^{(i)}$, and is randomly selected from the discrete interval $[0, m_i]$ according to a uniform distribution. The price is random but bound by the number of coins currently held.

The actors are adaptive in a weak sense: if they have less (resp. more) coins they probably will offer less (resp. more). Their changing circumstances are defined solely by how many coins they hold. The law of value, if it is to function, must therefore do so only via money flows.

$\mathbf{O_1}$ is one of many possible adaptive rules, but it is the simplest, and represents minimal theoretical commitment to the decision processes employed by actors in real economies. The aim is to concentrate modelling effort on the structural determinations of the conditions under which those evaluations take place, rather than the process of evaluation itself. Rule $\mathbf{O_1}$ assumes that, absent a decision theory, a range of possible decision outcomes are equally likely.

It may be that for the law of value to operate in general requires economic actors employ a form of rationality that is absent from rule $\mathbf{O_1}$. For example, $\mathbf{O_1}$ entails that an actor's money holding falls geometrically on average by half each transaction. Actors producing commodities with long production periods will encounter bankruptcy before the next sale. A more rational actor would plan their price offers to even out expenditures over the entire production period and also allocate expenditures to maximise personal utility, however measured. A deeper model would include this and would address to what extent political economy can be abstracted from micro-foundations.

4.6 The market

Periodically actors meet in the marketplace. Trading behaviour continues until the market is cleared when for every commodity type there are either no buyers or no sellers. Commodities are bought and sold in single units. A cleared market does not imply that all needs are satisfied or all commodities sold.

> Market clearing rule $\mathbf{M_1}$: (Stochastic). Initialise the set of uncleared commodities to $C = \{i : 1 \leq i \leq L\}$.
>
> 1. Randomly select an uncleared commodity i from the set C according to a uniform distribution.
> 2. Form the set of candidate sellers S, which contains all actors with a desire to sell commodity i (i.e., $S = \{x : \mathbf{e}_x[i] > \mathbf{d}_x[i], 1 \leq x \leq N\}$). Select the seller s from S according to a uniform distribution.
> 3. Form the set of candidate buyers B, which contains all actors with a desire to buy commodity i (i.e., $B = \{x : \mathbf{d}_x[i] > \mathbf{e}_x[i], 1 \leq x \leq N\}$). Select the buyer b from B according to a uniform distribution.
> 4. If no seller or no buyer (i.e., $S = \emptyset \vee B = \emptyset$) then remove commodity i from C; otherwise, invoke market exchange rule $\mathbf{E_1}$ (see below).
> 5. Repeat until there are no remaining uncleared commodities (i.e., $C = \emptyset$).

Rule $\mathbf{M_1}$ matches buyers with sellers who then conditionally exchange coins for commodities according to:

> Market exchange rule $\mathbf{E_1}$: (Stochastic). Given a buyer b and seller s of commodity i with offer prices $k_i^{(b)}$ and $k_i^{(s)}$ respectively, determined by price offer rule $\mathbf{O_1}$, select the exchange price, x_i, from the discrete interval $[k_i^{(b)}, k_i^{(s)}]$ according to a uniform distribution. The exchange price is randomly selected to lie between the two offer prices.
>
> If the buyer has sufficient funds ($m_b \geq x_i$) then the transaction takes place. Actor b loses x_i coins and gains one unit of commodity i, and the appropriate element of its endowment vector, $\mathbf{e}_b[i]$, is incremented by one. Actor c gains x_i coins and loses one unit of commodity i, and the appropriate element of its endowment vector, $\mathbf{e}_s[i]$, is decremented by one.

Rules $\mathbf{M_1}$ and $\mathbf{E_1}$ do not represent a typical Walrasian market in which transactions take place at equilibrium after a process of extended price signalling or 'tatonnement'. Instead, transactions occur at disequilibria prices, commodities may go unsold, and the same commodity type may exchange for many different prices in the same market period. Further, commodities in oversupply may initially fail to sell only to find willing buyers at a later time, and commodities in undersupply may not necessarily realise a higher price. In sum, although the rules do implement short-term price signalling due to disequilibrium between supply and demand the detailed dynamics of this process are not straightforward, and can only be approximated by mathematical models that assume continuous and immediate price adjustment.

4.7 Division of labour

The set $A_i = \{j : 1 \leq j \leq N, A(j) = i\}$ contains those actors that specialise in the production of i. The set $D = \{A_i : i = 1, \ldots, L\}$ partitions the actors into production sectors and represents the total division of labour of the economy. The division of labour is dynamic because actors can change what they produce. Actors attempt to meet their consumption requirements but do not explicitly maximise wealth. They switch from one production sector to another according to the following rule:

> Sector-switching rule $\mathbf{S_1}$: (Stochastic). For actor i at the end of every jth period of length T time steps form the consumption error, defined as the Euclidean norm of the consumption deficit vector, $\|\mathbf{d}_i^{(j)}\|$. $\|\mathbf{d}_i^{(j)}\|$ is compared to the consumption error of the previous period $\|\mathbf{d}_i^{(j-1)}\|$. If $\|\mathbf{d}_i^{(j)}\| > \|\mathbf{d}_i^{(j-1)}\|$ then randomly select a new production sector from the available L according to a uniform distribution. In other words, if the consumption error has increased from the previous period then swap to a new sector.
>
> T is a constant multiple of the maximum consumption period, $\max(c_i)$, such that actors produce and have the opportunity to sell at least one commodity before sampling the consumption error and deciding whether to switch.

There are no switching costs. The result of all actors following rule $\mathbf{S_1}$ is to perform a parallel search over possible social divisions of labour. Dissatisfied actors randomly switch to new sectors in search of sufficient income to meet their consumption requirements.

4.8 Simulator

The cycle of production, consumption, exchange and reallocation of social labour proceeds according to the following rule:

> Simulation rule $\mathbf{R_1}$: Randomly construct production ($\mathbf{p}$) and consumption vectors ($\mathbf{c}$) for the economy, such that the reproduction coefficient $\eta = 1$. Allocate M/N coins to each of the N actors.
>
> 1. Increment the global time step.
> 2. For each actor invoke production rule $\mathbf{P_1}$.
> 3. For each actor invoke consumption rule $\mathbf{C_1}$.
> 4. Invoke market clearing rule $\mathbf{M_1}$.
> 5. For each actor invoke sector-switching rule $\mathbf{S_1}$.
> 6. Repeat.

The rule set for the simple commodity economy

$$\mathbf{SCE} = \{\mathbf{R_1}, \mathbf{P_1}, \mathbf{C_1}, \mathbf{O_1}, \{\mathbf{M_1}, \mathbf{E_1}\}, \mathbf{S_1}, \}$$

defines the computational model. The implementation has five parameters: (i) the number of actors N, (ii) the number of commodities L, (iii) the amount of coins in the economy M, (iv) an upper bound, R, on the maximum possible consumption period, which is used to constrain the random construction of production and consumption vectors during initialisation, and (iv) a switching parameter C that is the constant multiple of the maximum consumption period required by sector-switching rule $\mathbf{S_1}$.

5 Simulation results

An execution of the computational model is a single sample of its parameter space. It isn't practical to explore the entire parameter space because computational resources are limited. The sampling process is therefore biased toward subspaces that can be feasibly computed. Some samples of the feasible parameter space are degenerate, in the sense that the law of value could not operate, for example economies with a single coin. Other degenerate settings are less obvious: for example, if the consumption period of a commodity i greatly exceeds the number of actors (i.e., if $R \gg N$) then the probability that a seller of i will find a buyer in the marketplace is low. An economy with insufficient actors will fail to support high frequency market activity, and exchange becomes occasional, failing a requirement for the law of value to operate. Results therefore are sampled from the feasible, non-degenerate parameter space.

All simulation runs follow a similar pattern of initial non-equilibrium activity prior to settling down to stable averages and stationary distributions (appendix B contains further experimental details). Many variables of interest could be measured but here the focus is the stationary distributions of the division of labour and market prices.

5.1 Division of labour

The distribution of actors in each sector of the economy settles to a normal distribution centred on a mean sector size. Figure 1 shows the evolution of the mean number of actors employed in each sector and the stationary distributions in a typical run. The equilibrium mean size of sector i is always approximately $N\frac{p_i}{c_i}$. Figure 2 reveals this relationship sampled over many runs.

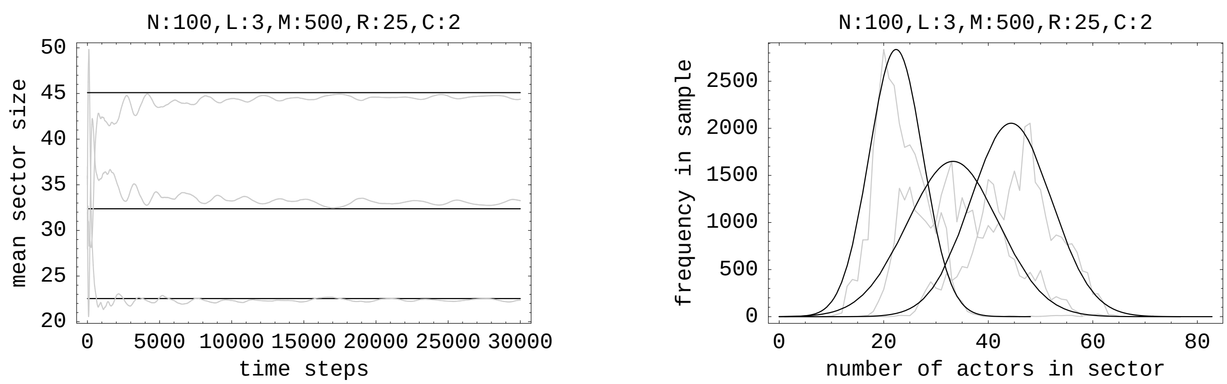

Figure 1: Evolution of mean sector size in a 3-commodity economy. The straight lines represent efficient sector sizes (left figure). Stationary distributions of sector sizes with fitted normal distributions (right figure).

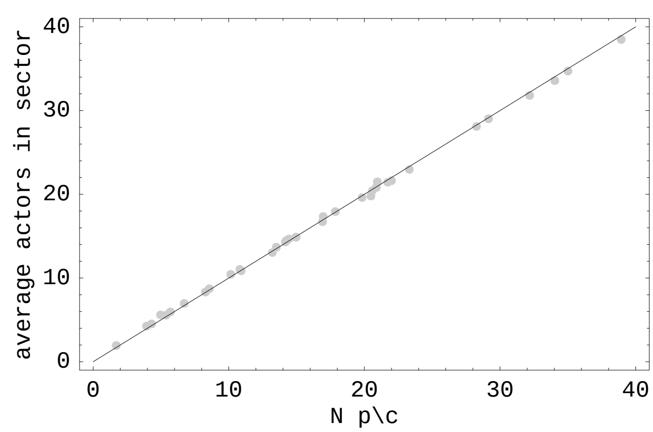

Figure 2: Relationship between mean sector size and $N\frac{p_i}{c_i}$ from 20 random samples of 3-commodity economies with parameter settings N:50,L:3,M:50,R:25,C:2. The straight line represents the identity relationship $y = x$.

Definition 1. A division of labour is *efficient* if for every commodity type the number of of commodities produced equals the social demand.

Proposition 1. Let $a_i = |A_i|/|D|$ be the proportion of actors producing commodity i. Then $a_i = \frac{p_i}{c_i}$ ($i = 1, \ldots, L$) is an efficient division of labour.

Proof. The social demand for commodity i is $\frac{N}{c_i}$ units per unit time. When $a_i = \frac{p_i}{c_i}$ the number of units produced is $N \frac{a_i}{p_i} = \frac{N}{c_i}$ units per unit time, which equals the social demand. □

On average the division of labour is approximately efficient, but due to stochastic effects perfect efficiency is never achieved. An efficient division of labour implies that the global consumption error is minimised and welfare maximised, absent market friction. Actual runs only approximate maximum welfare, and unsold commodities and unsatisfied demands either stabilise or slowly accumulate over time.

The results confirm that the **SCE** attains a (dynamic) equilibrium of the division of labour, and that the labour equilibrium is approximately efficient.

5.2 Objective prices

The stationary distribution of a commodity's price is exponential. Figure 3 shows the evolution of mean prices during a typical run and the associated stationary distributions. The price distributions have an exponential tail at the high end, but drop to zero at the low end, but the exponential distribution accurately models the price distributions over most of the price range. In equilibrium a single commodity type does not have a single price, but has a range of prices that occur with differing but fixed probabilities.

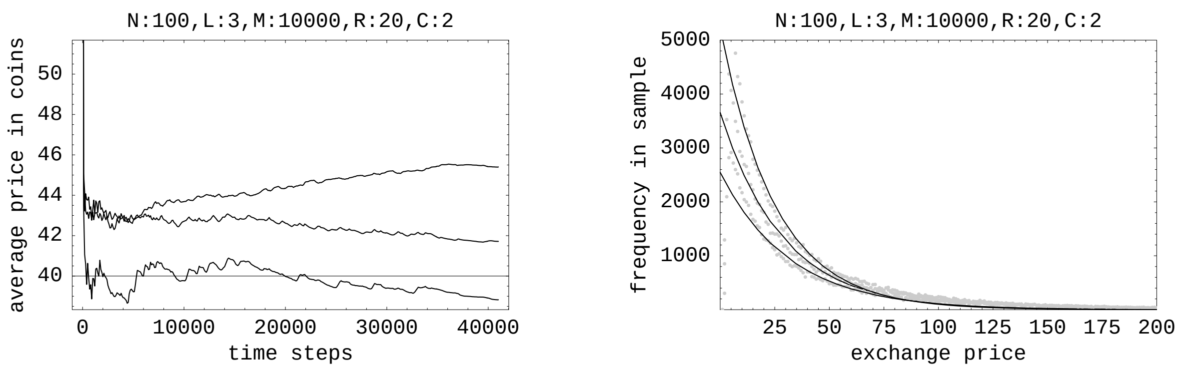

Figure 3: Evolution of mean commodity prices in a 3-commodity economy (left figure) and stationary distribution of commodity prices with fitted exponential distributions (right figure).

The law of value states that, in equilibrium, market prices 'correspond' to labour values. The Pearson correlation coefficient between two vectors, $\mathbf{x}$ and $\mathbf{y}$, given by

$$r_{xy} = \frac{\sum_i (x_i - \bar{\mathbf{x}})(y_i - \bar{\mathbf{y}})}{\sqrt{\sum_i (x_i - \bar{\mathbf{x}})^2} \sqrt{\sum_i (y_i - \bar{\mathbf{y}})^2}}$$

measures the linear relationship between them ($-1 \leq r \leq 1$). A value of -1.0 is a perfect negative (inverse) correlation, 0.0 is no correlation, and 1.0 is a perfect positive correlation. $r = 1.0$ implies that there is a single scalar constant such that $\mathbf{x} = \alpha \mathbf{y}$. Correlation is employed to measure the correspondence between market prices and labour values. Denote the average price of commodity i as $\langle k_i \rangle$. Figure 4 graphs representative time series of the correlation between the market price vector $\mathbf{k} = (\langle k_1 \rangle, \ldots, \langle k_L \rangle)$ and the labour values vector $\mathbf{l} = (p_1, \ldots, p_L)$ (recall that p_i is the time period required to produce commodity i). The main empirical result of this paper can now be stated: the correlation between mean market prices and labour values approaches unity in equilibrium. Table 1 contains further experimental results that demonstrate the robustness of this result.

The results confirm that the **SCE** attains a (dynamic) equilibrium in which the mean equilibrium price of a commodity, measured over a sampling period, is proportional to the labour-time required to make it.

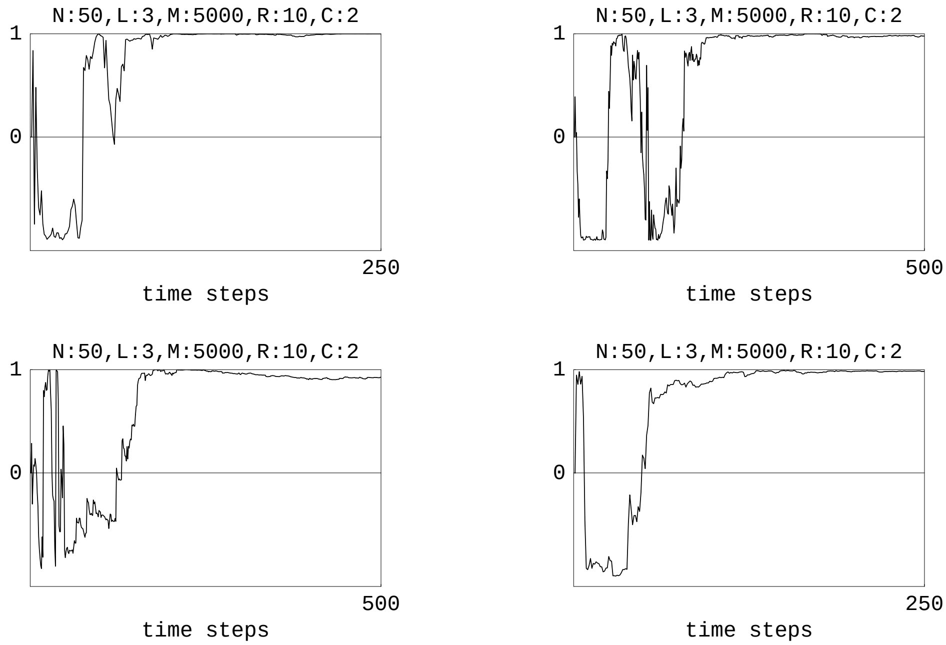

Figure 4: Evolution of vector correlation of mean prices and labour values over 4 runs of 3-commodity economies

Prices ‘gravitate’ around labour values and this equilibrium coexists with local and subjective pricing decisions constrained only by money endowments.

The equilibrium constant of proportionality, λ, between mean prices and labour values, such that $\mathbf{k} \approx \lambda \mathbf{l}$, must have dimensions *coins per unit labour-time*. λ summarises the causal relationship between expenditure of labour-time in production and the representation of that time in the market price of commodities. It measures how much labour-time money represents. Duménil (1983) and Foley (1982) first defined this constant and emphasised its importance for capitalist economies.

Definition 2. The *Monetary Expression of Labour-time* (MEL) is the ratio of the net product at current prices relative to the productive labour expended in an economy over a given period of time.

The MEL can be directly measured in the **SCE** and is given by

$$\lambda = \frac{\gamma M}{\sum_{i=1}^{L} p_i v_i} \tag{1}$$

where γ is the proportion of the total money in the economy that on average exchanges per unit time (hence γM is the mean money velocity in the economy), and v_i is the mean exchange velocity of commodity i. The numerator in the definition is the rate of money exchange, the denominator is the rate of labour-time exchanged in the form of commodities, and the MEL is the ratio of the two, measured in coins per unit of labour-time.

Figure 5 plots equilibrium mean prices, $\langle k_i \rangle$, against labour values multiplied by the MEL, λp_i, for a typical run of a 10-commodity economy (the largest that may be reasonably simulated). It demonstrates that the MEL is the constant of proportionality implied by the correlation results. The role of money as a representation of labour-time is particularly clear in this relationship.

The definition of MEL does not represent a causal theory of how the MEL is determined. The value of MEL will vary under different ‘institutional’ arrangements, such as how the market operates in detail, what kind of money and commodity throughput obtains, and so forth. Unlike the venerable quantity theory of money $MV = PT$ (where M is money, V is money velocity, P is the price level, and T the level of transactions), which is an accounting identity between market phenomena, the MEL abstracts a non-obvious causal relationship between non-market phenomena (production times) and market phenomena (prices). The emergence of MEL in this simple model should lend weight to arguments in favour of its applicability to real capitalist economies.

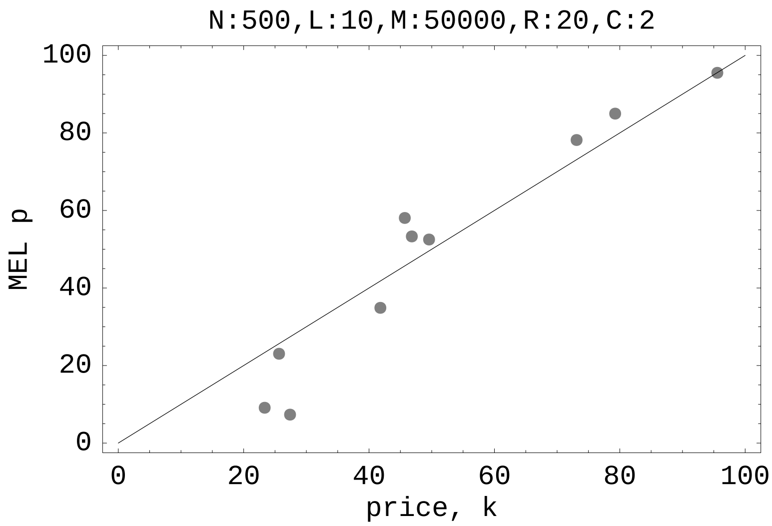

Figure 5: Stationary market prices and MEL transformed labour values in a 10-commodity economy with $r_{kl} = 0.96$. The straight line represents the identity relationship $y = x$.

As an aside, the quantity equation of money, $MV = PT$, holds in the **SCE**, as expected (i.e. $MV = PT \equiv \gamma M = \sum_{i=1}^{L} \langle k_i \rangle v_i$). Further analysis of this relationship belongs to a theory of the determination of the MEL constant, which is not developed here. But briefly, in the **SCE** both the quantity of money can determine the price level (i.e. left to right causation in the quantity equation) and the production and consumption times can determine the quantity of money required to circulate commodities (i.e. right to left causation).

6 Analysis

The experimental results demonstrate that (i) a (dynamic) equilibrium is reached, in which (ii) mean prices are linearly related to labour values by a constant of proportionality called the monetary expression of labour-time (MEL), and (iii) social labour is allocated approximately efficiently. The computational model generates these empirical regularities, but does not provide an adequate explanation of them. The law of value *emerges* from dynamic interactions of the constituent parts of the **SCE**, but a theory is required to explain this emergence.

The qualitative theory of the law of value was most fully developed by Isaak Rubin in his 1928 book, 'Essays on Marx's Theory of Value'. Rubin's theory is extended by approximating the **SCE** with a system of continuous time, flow-based differential equations that refer to the means of the variables of interest. The aim is to explain the gross qualitative features of the **SCE** rather than develop an accurate stochastic theory of the empirical distributions. Nevertheless this simplified approach does allow the equilibrium price distributions to be recovered by maximising entropy at the fixed point of the resulting dynamical system.

6.1 The labour equation

The rate money enters and leaves the market, or money velocity, is a proportion of the total money in the economy, denoted γM $(0 \leq \gamma \leq 1)$. It is assumed that γ is fixed constant, which is an approximation. A money allocation vector, $\mathbf{b}(t) = (b_1, \ldots, b_L)$, where $\sum_{i=1}^{L} b_i = 1$ and $0 \leq b_i \leq 1$, represents the instantaneous proportion of the total money flow received by each sector at time t. The sectoral income rate is therefore given by $b_i \gamma M$.

The average labour allocation vector, $\mathbf{a}(t) = (a_1, \ldots, a_L)$, where $a_i = |A_i|/|D|$ (see section 4.7), $\sum_{i=1}^{L} a_i = 1$ and $0 \leq a_i \leq 1$, represents the proportions of actors 'employed' in each sector at time t.

The mean price of a commodity is used to approximate its price distribution. Recall that the average price of commodity i as $\langle k_i \rangle$. The average cost of the universal commodity bundle, given current prices, is then $\sum_{i=1}^{L} \langle k_i \rangle / c_i$.

Actors switch sectors based on the consumption error, which is a function of the quantities of commodities received. To simplify the analysis price signals, in the form of the mismatch between income and the average cost of the commodity bundle, are used as a proxy for the consumption error. This simplifying assumption is used in the remainder of the analysis.

Each sector has an ideal expenditure rate that represents the money that would need to be spent in order for the constituent actors to meet their desired consumption rates. The rate is a function of the number of actors in the sector and current prices, and is given by: $a_j N \sum_{i=1}^{L} \langle k_i \rangle / c_i$.

The sectoral income error, denoted ϕ_i, measured in coins per unit time, is the difference between the actual income rate and the ideal expenditure rate:

$$\phi_i(t) = b_i \gamma M - a_i N \sum_{i=1}^{L} \frac{\langle k_i \rangle}{c_i}$$

A value of $\phi_i > 0$ implies a sectoral 'profit' (the sector receives more income than its constituent actors require to purchase the commodity bundle), $\phi_i < 0$ implies a sectoral deficit (there is insufficient income for the actors employed in the sector to purchase the commodity bundle), and $\phi_j = 0$ implies sectoral income equals ideal expenditure.

The switching behaviour of actors is approximated by assuming that the rate of change of labour allocation (or sector size) is proportional to the sectoral income error:

$$\frac{d}{dt} a_i = \beta \phi_i(t) = \beta(b_i \gamma M - a_i N \sum_{i=1}^{L} \frac{\langle k_i \rangle}{c_i}) \tag{2}$$

where $\beta > 0$ is a reaction coefficient. It follows from the definition that $\phi_i < 0$ implies a net decrease in the sectoral population, and $\phi_i > 0$ a net increase, subject to the constraint $\sum_{i=1}^{L} a_i = 1$. (2) is termed the *labour equation* because it defines how the allocation of labour to different sectors of production changes according to the money income received from the sale of commodities. The labour equation for the whole economy in vector notation is:

$$\dot{\mathbf{a}} = \beta(\gamma M \mathbf{b} - N(\mathbf{k} \cdot \mathbf{c})\mathbf{a}) \tag{3}$$

where $\mathbf{k} \cdot \mathbf{c}$ is the dot product of the average price vector and the consumption vector. Equation (3) can be simplified by adopting Say's Law as an approximation, which assumes that all commodities are sold (even when in oversupply). This allows the average price of a commodity to be approximated by the current sectoral income rate divided by the sectoral production rate:

$$\langle k_i \rangle = \frac{\gamma M}{N} \frac{b_i}{a_i} p_i \tag{4}$$

Hence, $\langle k_i \rangle$ becomes a function of a_i and b_i.

6.2 The money equation

The labour equation describes how the division of labour depends on incomes and prices, but as yet there is no model of how these change. A sector's income depends on the number of commodities produced. The production rate for commodity i is given by $a_i N / p_i$ and the maximum possible social consumption rate or 'social demand' for commodity i is N / c_i.

The sectoral 'production error', denoted ξ_i, measured in units of commodity i per unit time, is the difference between supply and demand:

$$\xi_i(t) = \frac{a_i N}{p_i} - \frac{N}{c_i}$$

A value of $\xi_i > 0$ implies over-production, $\xi_i < 0$ implies under-production, and $\xi_j = 0$ implies supply equals social demand. It is assumed that market rule $\mathbf{M_1}$ operates such that it can be approximated by assuming that the expected relationship between supply, demand and price holds: commodities in over-supply have lower average prices than those in under-supply. This implies that the rate of change of sector income is negatively proportional to the production error:

$$\frac{d}{dt} b_i = -\alpha \xi_i(t) = -\alpha N(\frac{a_i}{p_i} - \frac{1}{c_i}) \tag{5}$$

where $\alpha > 0$ is a reaction coefficient. It follows from the definition that $\xi_j < 0$ implies an increase in sectoral income, and $\xi_j > 0$ a net decrease, subject to the constraint $\sum_{j=1}^{L} b_j = 1$. (5) is termed the *money equation* because it defines how the allocation of money to different sectors of production changes according to the over or under-production of commodities. The money equation for the whole economy in vector notation is:

$$\dot{\mathbf{b}} = -\alpha N(\mathbf{A}\mathbf{p}^T - \mathbf{c}) \tag{6}$$

where $\mathbf{A}$ is the L by L diagonal matrix with (i, i) entry equal to a_i and the (i, j) $(i \neq j)$ entry zero.

6.3 Equilibrium

The $2L$ labour (3) and money (6) equations mutually interact and describe the evolution of the division of labour via the mechanism of market price changes. The causal schema is as follows: (i) an existing division of labour results in (ii) over and under-production of commodities that causes (iii) error-correcting price changes on the market due to supply and demand, which (iv) generate changes in sectoral incomes that (v) cause actors that cannot meet their consumption requirements to swap sectors, resulting in (vi) a new division of labour. Some results are now derived that show that the mutual interaction results in an equilibrium point at which prices equal labour values.

Definition 3. A *simple commodity system* is described by the following system of $2L$ coupled differential equations:

$$\dot{\mathbf{a}} = \beta(\gamma M\mathbf{b} - N(\mathbf{k} \cdot \mathbf{c})\mathbf{a}) \tag{7}$$

$$\dot{\mathbf{b}} = -\alpha N(\mathbf{A}\mathbf{p}^T - \mathbf{c}) \tag{8}$$

where

$$\langle k_i \rangle = \frac{\gamma M}{N} \frac{b_i}{a_i} p_i$$

and $\mathbf{A}$ is the L by L diagonal matrix with (i, i) entry equal to a_i and the (i, j) $(i \neq j)$ entry zero, subject to the constraints

$$\begin{aligned}
&\sum_{i=1}^{L} a_i = 1, && 0 \leq a_i \leq 1 \\
&\sum_{i=1}^{L} b_i = 1, && 0 \leq b_i \leq 1 \\
&\sum_{i=1}^{L} \frac{p_i}{c_i} = 1 = \eta && p_i, c_i > 0 \\
&\alpha, \beta, M, N > 0 \\
&0 \leq \gamma \leq 1
\end{aligned}$$

Lemma 1 (Equilibrium point). The simple commodity system has the unique equilibrium point

$$\mathbf{a}^* = (\frac{p_1}{c_1}, \frac{p_2}{c_2}, \ldots, \frac{p_L}{c_L}) = \mathbf{b}^* \tag{9}$$

(The proof is in appendix A.)

The lemma states that $\dot{a_i} = \dot{b_i} = 0$ (i.e., the system is at rest) when the proportion of actors employed in a sector equals the proportion of money received by the sector, and that proportion is p_i/c_i. This makes intuitive sense: every actor consumes the same consumption bundle, therefore, on average, they require the same income (otherwise actors move to different sectors and the system is not at rest). The lemma does not imply that every actor receives the same income in equilibrium, only that sectoral averages are equal. (In fact, the stationary income distribution in the **SCE** is highly unequal and approximately exponential).

Lemma 2 (Global stability). The equilibrium point is globally asymptotically stable. (The proof is in appendix A.)

The lemma states that the system, regardless of its initial conditions, always approaches the equilibrium point. The simple commodity system is a feedback system that functions to minimise both income and production 'errors'. This formalises Rubin's assertion that '[a] given level of market prices, regulated by the law of value, presupposes a given distribution of social labour among the individual branches of production. ... Marx speaks of the "barometrical fluctuations of the market prices". This phenomenon must be supplemented. The fluctuations of market prices are in reality a barometer, an indicator of the process of distribution of social labour which takes place in the depths of the social economy. But it is a very unusual barometer; a barometer which not only indicates the weather, but also corrects it' (Rubin (1973, p.78)). Lemma 2 explains why simulation runs tend to equilibrium.

Theorem 3 (Efficient division of labour). The division of labour is efficient in equilibrium.

Proof. By lemma 1 the proportion of actors in sector i at equilibrium is $a_i = \frac{p_i}{c_i}$, which by proposition 1 is efficient. □

Theorem 3 explains why the simulation tends to an approximately efficient division of labour. The empirical results do not exhibit perfect efficiency because the **SCE** violates the simplifying assumptions of the mathematical model.

Theorem 4 (The law of value). Labour values are global attractors for average market prices.

$$\lim_{t \to \infty} \mathbf{k}(t) = \lambda \mathbf{l} \tag{10}$$

At equilibrium the average price of a commodity is proportional to the labour-time required to make it. The constant of proportionality, $\lambda = \gamma M/N$, represents the monetary value of one unit of labour-time.

Proof. Substituting the equilibrium point, $a_i = p_i/c_i = b_i$, into (4) yields $\langle k_i \rangle = \lambda p_i$, which by lemma 2 is the globally asymptotically stable market price. □

Theorem 4 accounts for the observed correlations between prices and labour values.

In equilibrium actors receive equal mean incomes but are engaged in productive activity of unequal periods. Hence, commodities that take longer to produce sell for higher mean market prices. This is the fundamental reason why prices correspond to labour values at the equilibrium of the simple commodity economy.

6.4 Disequilibrium deviation of price from value

A key insight of Marx's theory of the law of value is that prices *refer* to amounts of labour time and *deviations* of prices from values are social *error signals* that function to redistribute labour. Only when labour is efficiently distributed are prices proportional to labour values.

The deviation of price from value at disequilibria can be analysed by introducing the concept of essential price (Brinkman (2002)).

Definition 4. A commodity realises an *essential price* in exchange. It is measured in units of labour-time and is the money price divided by the MEL. The mean essential price

$$\langle \epsilon_i \rangle = \frac{\langle k_i \rangle}{\lambda} \tag{11}$$

represents how much social labour-time a commodity on average fetches in the marketplace. If a commodity type exchanges at essential price $\epsilon_i < p_i$ then it is *undervalued*, if exchanged at $\epsilon_i > p_i$ it is *overvalued.*

(In fact, 'essential price' is a synonym for Marx's 'exchange-value' but it has the advantage of emphasising that it is value in a 'money form').

At equilibrium $\langle \epsilon_i \rangle = p_i$ for all $i = 1, \ldots, L$ but otherwise commodities sell at overvalued or undervalued essential prices, in accordance with the laws of supply and demand.

An act of exchange involves more than swapping of a commodity for an amount of money. It is also an exchange of a representation of an amount of social labour-time, measured by the essential price, for an amount of private labour-time actually expended in the production of the commodity. Normally this is not an exchange of equivalents. If the global division of labour mismatches the social demand then labour associated with scarce commodities is rewarded with access to additional social labour-time, whereas labour associated with unwanted commodities is punished by a reduction of access. Out of equilibrium not all private labours are mutually equalised and not all private labours are socially necessary. But if the reallocation of labour resources is based on these monetary reward signals then the feedback loop completes and a division of labour emerges in which unnecessary private production is minimised and prices approach labour values. This role of essential price as regulator of the division of labour is apparent in the following important relationship

$$\dot{d}_i \propto a_i\big(\frac{\langle\epsilon_i\rangle}{p_i} - 1\big) \tag{12}$$

which is derived by substituting (11) into (2). The term in brackets is positive if the commodity type is overvalued (implying an increase in the sector size) and negative if the commodity type is undervalued (implying a decrease in the sector size). Equation (12) reveals the causal connection between labour allocation and price forms that occurs under the surface of the simple commodity economy. It is a precise formulation of Rubin's observation that 'value is the transmission belt that transfers the movement of working processes from one part of society to another, making that society a functioning whole' and summarises how the interaction of private commodity producers, using a monetary representation of the total social labour-time, spontaneously allocates labour to different branches of production according to social demand.

6.5 Stationary price distributions

Foley (1994) introduced a statistical equilibrium theory of markets, which are viewed as entropy maximising processes. The theory predicts that markets spontaneously generate horizontal inequality between market participants. Dragulescu and Yakovenko (2000) discuss a simulation of a simple money economy (no commodities and probabilistic transfer of money in exchange) that generates an exponential distribution for actor income. They compare conservation of kinetic energy during atomic collisions in a perfect gas and the conservation of money during exchanges in a closed economy and reason that the probability distribution of money should follow the Boltzmann-Gibbs law $P(m) = Ce^{-m/T}$ where T is an effective temperature equal to the average amount of money per economic agent. The probability that an actor will enjoy a sequence of highly advantageous trades is small and therefore a majority of low incomes coexist with an exponentially decreasing minority of very high incomes.

The **SCE** also generates similar income inequalities but the income distribution is only approximately exponential. Sector income distributions are heterogeneous because (i) actors in those sectors produce and sell commodities at different rates but (ii) spend on average half their money on each transaction (due to price offer rule $\mathbf{O_1}$). Hence, sectors with long (resp. short) production periods tend to contain actors with low (resp. high) mean incomes. Although mean sector incomes approach each other toward equilibrium, they do not reach equality. The overall income distribution therefore, which is the sum of the sector distributions, diverges from an exponential. But price distributions, specific to each sector, are more closely exponential (as discussed in section 5.2), and can be explained by a standard argument from statistical mechanics (e.g. see Wannier (1987)).

Proposition 2. The equilibrium probability distribution of commodity i realising price k_i is given by the exponential distribution

$$P(k_i) = \frac{1}{\lambda p_i} e^{-k_i/\lambda p_i} \tag{13}$$

Proof. The proof, given in appendix A, follows the derivation of the Boltzmann-Gibbs law under the assumption that the **SCE** is an entropy-maximising process. □

The mean value $\langle k_i\rangle$ of the price distribution is the MEL multiplied by the commodity's labour value, as required (i.e. $E(K) = \int_0^\infty k_i P(k_i)\mathrm{d}k_i = \lambda p_i$).

Figure 6 compares the predictive accuracy of (13) against data from a simulation run. The derivation of (13) assumes that all exchange prices are equally likely subject to mean price constraints. The assumption is violated by the **SCE** in part because mean sector incomes are not identical, which partially accounts for the deviation between theory and data.

The exponential price distribution will be sensitive to the particular price offer rule (or rules) employed by the actors. The more important point, therefore, is that in statistical equilibrium the same commodity type realises a range of different market prices, $k_i^{(1)}, k_i^{(2)}, \ldots$, each of which represents different transfers of social labour-time between buyer and seller. The role of essential price in regulating the division of labour is apparent 'on average' and is a property of the price distributions, not a property of individual transactions. Hence, a commodity type may be correctly valued in equilibrium while, at the same time, particular transactions may represent under or overvaluations of the commodity instance. The law of value states that, whatever the precise distribution of exchange prices, mean equilibrium prices are proportional to labour values. The view that the law of value manifests in probability distributions rather than individual transactions is in accordance with the probabilistic approach to political economy outlined by Farjoun and Machover (1989).

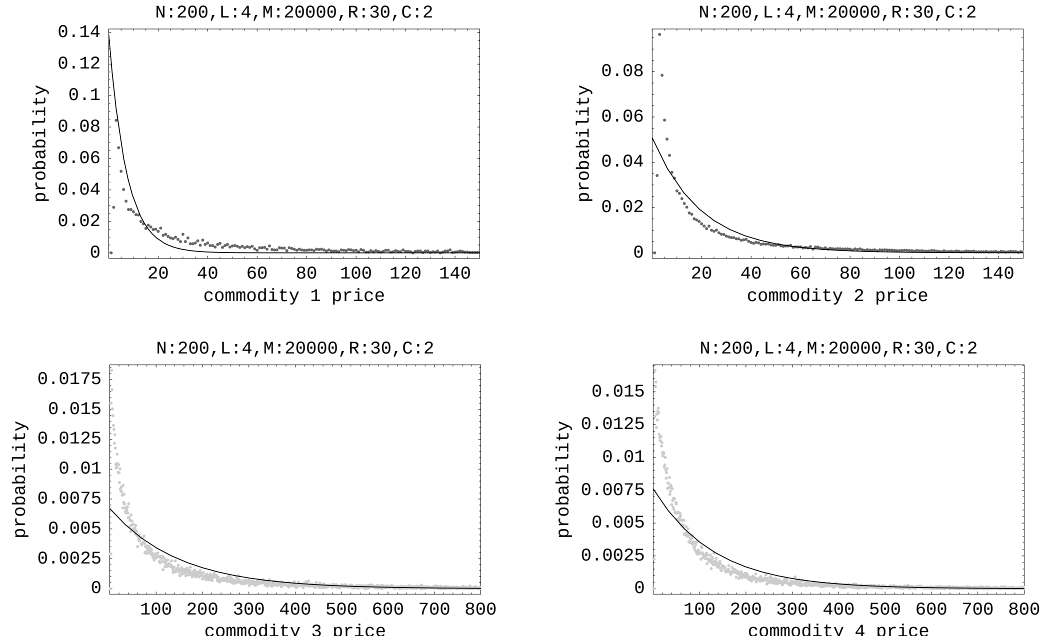

Figure 6: Actual (light plot) and predicted (dark plot) stationary price probability distributions in a 4-commodity economy. The production periods for each commodity are 0.213759, 0.583765, 4.42487, and 3.89982 units of time respectively.

7 Discussion

Included here are some further remarks on the model.

The choice of modelling symbolic money (e.g. paper or coins), which has nominal but no intrinsic value, rather than money in the form of a commodity such as gold, which has intrinsic value in virtue of the labour required for its production, differs from Marx's presentation but has the advantage of separating two definitions that may be easily conflated in his analysis of money (for a discussion, see Foley (1983)): (i) the 'value of money', which is the inverse of the MEL and is the labour-time represented by the monetary unit (e.g. 1 hour of social labour-time is represented by 1 coin), and (ii) the 'value of the money commodity', which is the amount of social labour-time required for the production of a unit of the money commodity (e.g. 1 ounce of gold requires 1 hour of social labour-time for its production). Marx emphasised that the 'value of money' must be causally constrained by the 'value of the money commodity' if paper money is to function as a symbol of value at all: 'Only in so far as paper money represents gold, which like all other commodities has a value, is it a symbol of value' (Marx (1972, p. 129). In this paper the relationship between an underlying money-commodity and its symbolic representation by either convertible or inconvertible symbolic money is ignored. The symbolic money that circulates in the simple commodity economy may be variously interpreted as representing amounts of gold, in accordance with Marx's view, or as directly representing amounts of social labour-time via a social convention backed by state control of the money supply (or perhaps a combination of these interpretations, as they are not mutually exclusive). The analysis therefore concentrates on the general function of money, as expressed by the the monetary expression of labour-time, rather than a particular form of money, such as a money-commodity that also happens to have an associated labour value.

Assuming a frictionless market, in which all commodities clear and all demands are satisfied, equilibrium commodity velocities equal N/c_i. Replacing v_i with N/c_i in equation (1) gives

$$\lambda = \frac{\gamma M}{N \sum_{i=1}^{L} \frac{p_i}{c_i}} = \frac{\gamma}{\eta} \frac{M}{N} = \gamma \frac{M}{N}$$

which shows that the MEL of the simple commodity system is a special case of the MEL for the **SCE**. The analysis assumed λ is constant. But in fact the exchange velocity of a commodity in the **SCE** has an upper bound of $N \min(a_i/p_i, 1/c_i)$, which is dependent on the current division of labour. Substitution into (1) gives

$$\lambda(t) \geq \frac{\gamma M}{N} \frac{1}{\sum_{i=1}^{L} \min(a_i, \frac{p_i}{c_i})}$$

The first fraction in this expression represents the average income rate (or average expenditure rate). The constant γ summarises the aggregate effect of the spending rules that actors follow (e.g., price offer rule $\mathbf{O_1}$). The denominator in the second fraction measures the proportion of labour-time that is socially necessary (i.e., expended on commodities that find buyers on the market). The MEL depends on both how actors spend incomes and the current commodity throughput. Assuming γ is constant an increase in throughput lowers the MEL (hence a coin represents more social labour-time), and a decrease in commodity throughput raises the MEL (hence a coin represents less social labour-time). In fact, during simulation runs, the MEL tends to decrease as the social division of labour approaches efficiency. These dynamic aspects of the MEL have not been explored (and may be treated seperately from price inflation or deflation due to changes in the money supply).

The **SCE** model and associated parameter settings define sufficient conditions for the operation of the law of value. Experimentation with alternatives to price offer rule $\mathbf{O_1}$ (or different collections of rules in the same economy) would help determine what kinds of individual rationality help or hinder its operation. For example, in some preliminary experiments with actors able to learn simple price models the stationary price distributions were normal, rather than exponential, but centred on mean prices as predicted by the law of value. Another way to explore this issue is to develop a distributed implementation of the **SCE** that allows humans to take the role of economic actors in a laboratory setting, perhaps with a global scorecard of consumption errors to motivate the participants. In this context, decision theory may have a role explaining systematic deviations of price from value in conditions of perfect price competition.

It would be worthwhile to study the effect of weakening some of the assumptions of the model, for example introducing individual production and consumption vectors corresponding to an economy with differences in skill and consumption tastes. The analysis presented here suggests that the law of value will fully manifest provided some weak assumptions on the vector distributions hold, for example that they satisfy the balance condition of a generalised reproduction coefficient for a heterogenous economy. However, this is an open question. Another possibility is to introduce technical change, which would wrap the law of value in a slower dynamic process that changes labour values over time. In this case, the price attractors of the simple commodity economy would themselves be subject to dynamics. However, it seems strained to investigate technical change in the simple commodity economy, rather than an economy with capital investment.

Finally, the mathematical approximation of the **SCE** exhibits cross-dual dynamics that are structurally similar to models of the formation of prices of production due to capital mobility (e.g. Boggio (1985), Flaschel and Semmler (1985), and Duménil and Lévy (1985)). In both cases the allocation of a productive resource (i.e. labour or capital) to different industries is a function of a profit measure (i.e. the income error or profit rate) subject to competitive supply and demand dynamics. These models are potentially relevant to understanding extensions of the **SCE** model that include production of commodities by means of commodities, wage-labour, capital and profit-maximising behaviour.

8 Conclusion

The law of value emerges from the dynamic interactions of private commodity producers. In a simple commodity economy (i) labour values are global attractors for market prices, (ii) market prices are error signals that function to allocate the available social labour between sectors of production, and (iii) the tendency of prices to approach labour values is the monetary expression of the tendency to efficiently allocate social labour. The constant of proportionality of the linear relationship between labour values and market prices is the monetary expression of labour-time (MEL), which measures how many units of money represent one unit of social labour-time. The MEL summarises a non-obvious causal relationship between non-market phenomena (production times) and market phenomena (prices), and links the total available social labour-time to its monetary representation.

The concept of essential price, which measures how much social labour-time a commodity fetches in the marketplace, is important for theorising how deviations of price from value are labour re-allocation 'signals'. The essential price of a commodity normally mismatches the private labour-time expended in its construction, indirectly signalling whether the labour was socially necessary or not. The law of value operates 'behind the backs' of actors via money flows that place income constraints on their local evaluations of commodity prices.

The computational approach can avoid many of the unrealistic assumptions of traditional Walrasian models. The introduction of time, trade at disequilibrium prices, and markets that do not clear is a step towards reality.

The equilibrium of the simple commodity economy is a statistical equilibrium, in which a single commodity type may realise many different prices. In consequence, the regulating role of essential price is a property of price distributions, not individual transactions. Further, the law of value can only emerge in broad models of economic systems that complete the feedback loop between production, consumption, exchange *and* reallocation of labour resources.

An actor engaged in free exchange derives personal benefit from transactions and the immediate apprehension of this fact may motivate subjective theories of value. But an exchange has causal consequences beyond the immediate moment and the satisfactions of mutual commerce that derive from its embodiment within a system of generalised commodity production. Actors do not normally think money into existence although they do decide to spend more or less of what they have. Their income is a local representation of a global resource constraint not under their subjective control. Although money *flows* according to demands for use-values, and is normally accompanied by the satisfaction of desires, it *refers* to amounts of social labour-time. Local flows are easier to apprehend than global reference, which may partially account for the relative neglect of objective theories of value.

A. Proofs

Proof of lemma 1. The system of equations (7) and (8) can be treated as $2L$ pairs of simultaneous linear differential equations. Substituting (4) into (7) and considering a single sector:

$$\dot{a_i} = \beta\gamma M(b_i - a_i \sum_{j=1}^{L} \frac{p_j b_j}{c_j a_j}) \tag{14}$$

The non-linear sum in (14) can be eliminated as follows. Summing over all sectors:

$$\sum_{i=1}^{L} \dot{a_i} = \sum_{i=1}^{L} [\beta\gamma M(b_i - a_i \sum_{j=1}^{L} \frac{p_j}{c_j} \frac{b_j}{a_j})]$$

but given that

$$\sum_{i=1}^{L} a_i = 1 \implies \dot{a_1} + \dot{a_2} + \ldots + \dot{a_L} = 0$$

then

$$\sum_{i=1}^{L} [\beta\gamma M(b_i - a_i \sum_{j=1}^{L} \frac{p_j}{c_j} \frac{b_j}{a_j})] = 0 \implies \beta\gamma M \sum_{i=1}^{L} [b_i - a_i \sum_{j=1}^{L} \frac{p_j}{c_j} \frac{b_j}{a_j}] = 0$$

As $\beta\gamma M \neq 0$ then

$$\sum_{i=1}^{L} b_i = \sum_{i=1}^{L} a_i \sum_{i=1}^{L} \frac{p_i}{c_i} \frac{b_i}{a_i}$$

Recalling that $\sum_{i=1}^{L} a_i = 1$ and $\sum_{i=1}^{L} b_i = 1$ then

$$\sum_{i=1}^{L} \frac{p_i}{c_i} \frac{b_i}{a_i} = 1$$

Substitution into (14) yields a linear form of the labour equation:

$$\dot{a_i} = \beta\gamma M(b_i - a_i) \tag{15}$$

that is coupled with:

$$\dot{b_i} = -\alpha N(\frac{a_i}{p_i} - \frac{1}{c_i}) \tag{16}$$

Setting $\dot{\mathbf{a}} = \dot{\mathbf{b}} = \mathbf{0}$ yields the unique equilibrium point of the system. (16) implies $a_i = p_i/c_i$ and (15) implies $a_i = b_i$. This solution is valid and unique for economies with reproduction coefficient $\eta = 1$, such that the equalities $\sum_{i=1}^{L} a_i = \sum_{i=1}^{L} b_i = \sum_{i=1}^{L} p_i/c_i = 1 = \eta$ hold. □

Proof of lemma 2. A change of variables, $x_i = a_i - \frac{p_i}{c_i}$ and $y_i = b_i - \frac{p_i}{c_i}$ translates the equilibrium point to the origin. Given that $\dot{\mathbf{x}} = \dot{\mathbf{a}}$ and $\dot{\mathbf{y}} = \dot{\mathbf{b}}$ the transformed linear system is:

$$\dot{\mathbf{x}} = \beta\gamma M(\mathbf{y} - \mathbf{x})$$
$$\dot{\mathbf{y}} = -\alpha N \mathbf{X}\mathbf{p}$$

where $\mathbf{X}$ is the L by L diagonal matrix with (i,i) entry equal to x_i and the (i,j) $(i \neq j)$ entry zero.

The x_i and y_i represent production and income errors respectively. Consider the function

$$V : \mathbb{R}^{2L} \to \mathbb{R}$$
$$V(x_1, \ldots, x_L, y_1, \ldots, y_L) = \frac{1}{2\beta\gamma M}\sum_{i=1}^{L} x_i^2 + \frac{1}{2\alpha N}\sum_{i=1}^{L} p_i y_i^2$$

that associates a scalar error measure with each possible state of the simple commodity system. In fact, V defines an error potential.

Global stability is now deduced by Lyapunov's direct method (e.g. see Brauer and Nohel (1989)). V is positive definite as $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$. Hence, V is a Lyapunov function. V is now shown to be strictly decreasing on all state trajectories:

$$\begin{aligned} V^* &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + \ldots + \frac{\partial V}{\partial x_L}\dot{x_L} + \frac{\partial V}{\partial y_1}\dot{y_1} + \frac{\partial V}{\partial y_2}\dot{y_2} + \ldots + \frac{\partial V}{\partial y_L}\dot{y_L} \\ &= \frac{1}{\beta\gamma M}\sum_{i=1}^{L} x_i\dot{x_i} + \frac{1}{\alpha N}\sum_{i=1}^{L} p_i y_i \dot{y_i} \end{aligned}$$

Substituting for $\dot{x_i}$ and $\dot{y_i}$ gives

$$\begin{aligned} V^* &= \sum_{i=1}^{L} x_i(y_i - x_i) - \sum_{i=1}^{L} x_i y_i \\ &= \sum_{i=1}^{L} x_i y_i - \sum_{i=1}^{L} x_i^2 - \sum_{i=1}^{L} x_i y_i \\ &= -\sum_{i=1}^{L} x_i^2 \\ &\leq 0 \end{aligned}$$

with $V^* = 0$ only when $\mathbf{x}^* = \mathbf{0}$. In other words, as time progresses the simple commodity system always follows an error-reducing trajectory that approaches the origin. By Lyapunov's Theorem the equilibrium point is asymptotically stable. Stability properties for linear systems are global. Therefore the equilibrium point is globally asymptotically stable. □

Proof of proposition 2. The proof follows Boltzmann's original argument with appropriate modifications.

An actor in sector i that realises price k_i for commodity i has an income rate $\frac{k_i}{p_i}$, where p_i is the production period. Divide sector i's income rate axis, $0 \leq \frac{k_b}{p_i} \leq \infty$, into small bins of size db. Label each bin consecutively $b = 1, 2, \ldots$ Denote the number of actors in bin b as N_b. From theorem 3 the total number of actors in equilibrium is $\bar{N} = \frac{p_i}{c_i}N$.

The set $\{N_b\}$ denotes a possible set of occupancy numbers for the income bins. Other things being equal, the number of ways $\bar{N}$ can be distributed over the bins to generate the set $\{N_b\}$ is given by the multinomial distribution $\mathbf{M} = \bar{N}!/\prod_{b=1}^{\infty} N_b!$ and is proportional to the probability of the occurrence of set $\{N_b\}$ (e.g. see Wannier (1987)). Maximising the entropy of the occupancy distribution will give the probability that an actor has income rate of $\frac{k_i}{p_i}$ in equilibrium. The probability of an actor realising a particular price follows immediately from the probability of a particular income rate as p_i is common to all actors in the sector.

The logarithm of probability is the entropy

$$\ln \mathbf{M} = \ln \bar{N}! - \sum_{b=1}^{\infty} \ln N_b!$$

Using Stirling's approximation $\ln a! \approx a \ln a - a$ for the factorials and rearranging

$$
\begin{aligned}
&= \bar{N} \ln \bar{N} - \sum_{b=1}^{\infty} N_b \ln N_b \\
&= \bar{N}(\sum_{b=1}^{\infty} \frac{N_b}{\bar{N}} \ln \bar{N} - \frac{1}{\bar{N}} \sum_{b=1}^{\infty} N_b \ln N_b) \\
&= \bar{N}\big(\sum_{b=1}^{\infty} \frac{N_b}{\bar{N}} (\ln \bar{N} - \ln N_b)\big) \\
&= -\bar{N}(\sum_{b=1}^{\infty} \frac{N_b}{\bar{N}} \ln \frac{N_b}{\bar{N}})
\end{aligned}
$$

Hence the entropy per agent is

$$H(P_b) = -\sum_{b=1}^{\infty} P_b \ln P_b$$

where $P_b = N_b/\bar{N}$ is the probability that an actor has income rate k_b/p_i.

The maximum entropy distribution of occupancy numbers is found by solving the optimisation program

$$\max_{\{P_b\}} H(P_b) = -\sum_{b=1}^{\infty} P_b \ln P_b$$

subject to

$$\sum_{b=1}^{\infty} P_b = 1 \tag{17}$$

and

$$K = \sum_{b=1}^{\infty} \frac{k_b}{p_i} N_b = \frac{\bar{N}}{p_i} \sum_{b=1}^{\infty} P_b k_b \tag{18}$$

where constraint (17) ensures that probabilities sum to 1 and constraint (18) ensures that the sum of all individual income rates equals the total sector income rate, which from from lemma 1 is $K = \frac{p_i}{c_i}\gamma M$. The Lagrangian is

$$L \equiv -\sum_{b=1}^{\infty} P_b \ln P_b - (\alpha - 1)(\sum_{b=1}^{\infty} -1) - \beta(\sum_{b=1}^{\infty} P_b k_b - \frac{p_i K}{N})$$

with multipliers α and β. Setting $\frac{\partial L}{\partial P_b} = 0$ yields the Gibbs distribution law

$$
\begin{aligned}
&- \ln P_b - \alpha - \beta k_b = 0 \\
&\implies P_b = \frac{1}{Z(\beta)} e^{-\beta k_b}
\end{aligned}
$$

where $Z(\beta) = \sum_{b=1}^{\infty} e^{-\beta k_b}$ normalises the distribution and is known as the partition function (e.g. see Kapur (1989)). To determine β explicitly requires solving constraint (18):

$$\frac{1}{Z(\beta)} \sum_{b=1}^{\infty} k_b e^{-\beta k_b} = p_i \frac{K}{\bar{N}} \tag{19}$$

Assuming the bin size db is small and the k_b can be reordered to approximate a continuous distribution then

$$Z(\beta) \approx \int_0^{\infty} e^{-\beta k} dk = \frac{1}{\beta} \tag{20}$$

Similarly

$$\sum_{b=1}^{\infty} k_b e^{-\beta k_b} \approx \int_0^{\infty} k e^{-\beta k} dk = \frac{1}{\beta^2} \tag{21}$$

Substituting (20) and (21) into (19) yields

$$\beta = \frac{\bar{N}}{p_i K}$$

Substituting this value for β into the Gibbs distribution law (and recalling that $\bar{N} = \frac{p_i}{c_i} N$ and $K = \frac{p_i}{c_i}\gamma M$) yields the price distribution

$$P(k_i) = \frac{1}{\lambda p_i} e^{-k_i/\lambda p_i} \tag{22}$$

for commodity i, where $\lambda = \gamma M/N$ is the MEL. □

B. Experimental details

The **SCE** is defined to have reached a state of statistical equilibrium when the rate of change (sampled every 1000 time steps) of the labour value/market price vector correlation is lower than a small threshold. When this convergence condition is met the simulation continues for a further 5000 time steps in order to sample the stationary distributions. (An alternative convergence condition is to check when the rate of change of entropy of every commodity price distribution is lower than a small threshold, but this was not tried). An upper-limit of 200000 time steps is set in case convergence is not achieved within a reasonable time period. In almost every cases convergence is reached before the upper-limit. Market clearing rule $\mathbf{M}_1$ cycles until there are either no buyers or no sellers for every commodity. With a large number of actors the clearing loop takes a prohibitively long time, therefore, in practice, an upper limit of the maximum number of transaction attempts per actor is set. Once the number of maximum transactions is reached the actor is neither a buyer nor seller for any commodity. This can be interpreted as a 'time limit' on the market period.

L	3	4	5	6	7	8	9	10
corr.	1.0	0.99	0.98	0.96	0.98	0.88	0.96	0.96
	0.99	1.0	0.99	0.97	0.98	0.97	0.96	0.99
	0.98	0.94	0.99	0.99	0.97	0.94	0.96	0.96
	0.98	0.99	0.93	0.99	0.95	0.97	0.97	0.91
	0.99	0.99	0.99	0.99	0.94	0.91	0.86	0.92
	0.96	0.84	0.99	0.93	0.98	0.99	0.95	0.95
	1.0	0.95	0.99	0.99	0.96	0.93	0.95	0.98
	0.99	0.97	1.0	0.98	0.96	0.94	0.94	0.95
	1.0	0.96	0.96	0.95	0.95	0.94	0.93	0.99
	1.0	1.0	0.95	0.97	0.95	0.95	0.99	0.93
mean	**0.99**	**0.96**	**0.98**	**0.97**	**0.96**	**0.94**	**0.95**	**0.95**

Table 1: Labour value/market price correlations from random samples of the **SCE**, with parameter settings N:200, L:n ($n = 3, \ldots, 10$), M:500, R:20, C:2. Each parameter setting is sampled 10 times. Results are rounded to 2 decimal places. The current implementation runs out of memory when the number of commodities exceeds 10 (and is also prohibitively slow). If $L \to N$ (i.e. the number of commodities approaches the number of actors) then the economy is unlikely to sustain production rates and correlations will decrease.

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